Distributed Scalar Quantizers for Subband Allocation

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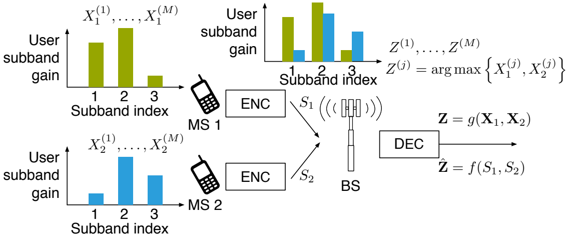
Outline

Motivation

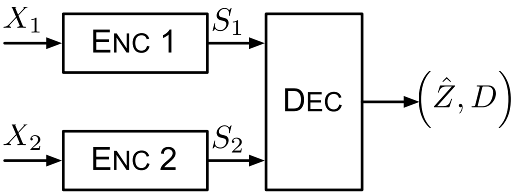
- Subbands in an OFDMA system must be assigned to a unique MS
- AMC: BS wants MS w/ best channel & the gain on that channel
- *Rateless codes* (e.g., ARQ): BS wants MS w/ best channel

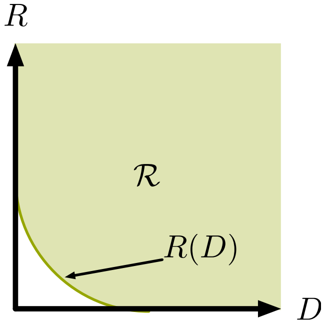


- BS does not need to reproduce MS local state
- Trade-off in feedback overhead & system efficiency

Related Work

CEO—Indirect Distributed Lossy Source Coding





- $S_i \in \{1, \ldots, 2^{nR_i}\}$
- $Z = g(X_1, X_2)$, $\hat{Z} = f(S_1, S_2)$
- $D = \mathbb{E}\left[d(Z, \hat{Z})\right]$, $R = R_1 + R_2$

- $\mathcal{R}$ achievable (R, D) pairs
- $R(D)$ smallest R s.t. $(R, D) \in \mathcal{R}$

Cover & Thomas 2006 ([1]) and El Gamal & Kim 2011 ([2])

Related Work

Distributed Functional SQ & Layered Architectures

Zamir & Berger (1999) [3]

- Lossy, continous sources
- Lattice quantizers w/ Slepian-Wolf (SW) coding
- Optimal asymptotically in rate

Servetto (2005) [4]

- Lossy, discrete sources
- Scalar quantizer w/ SW coding
- Optimal for all rates

Wagner et al. (2008) [5]

- Lossy (MSE),Gaussian sources
- Vector quantizers w/ SW coding
- Optimal for all rates

Misra et al. (2011) [6]

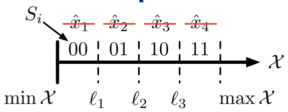
- Lossy (MSE), function of sources
- High rate regime
- Optimal asymptotically in rate

"Layered" Achievable Scheme

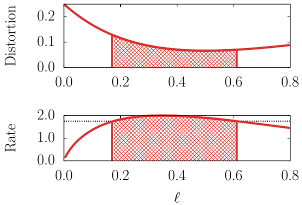
Scalar Quantizers at each user followed by entropy coding

Summary of Contributions

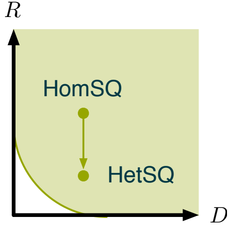
1. Distortion optimal HomSQs



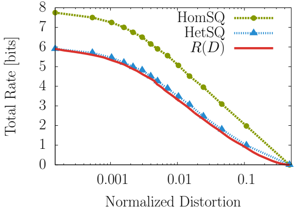
2. Entropy-constrained HomSQs



3. HetSQs superior to HomSQs for i.i.d. sources

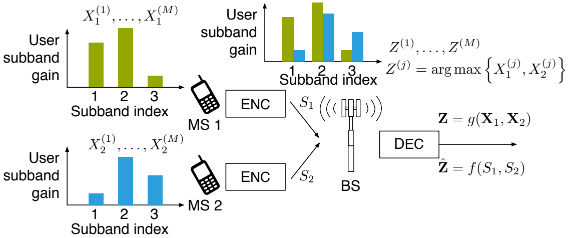


4. HetSQ can be close to fundamental limit



Outline

Problem Model & Notation



- X_i i.i.d. chan. capacity for MS i
- Z optimal subband allocation

$$Z = \arg\max_i \{X_i : i = 1, 2\}$$

- $\hat{Z}$ estimated subband allocation
- S_i coded message from MS i
- R_i rate achieved by MS i
- d distortion

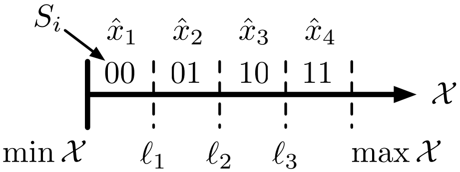
$$d(Z, \hat{Z}) = X_Z - X_{\hat{Z}}$$

We focus on two users w/ single subband

Outline

Scalar Quantizers

- K-bin SQ parameterized by
  - *decision boundaries* ℓ_i
  - *reconstruction points* $\hat{x}_i$
- Designed to meet distortion and/or rate constraints
- **Encoding:** report the index of the bin containing X_i
- **Decoding:** map bin index to reconstruction points



$\ell_0 \triangleq \min \mathcal{X}$ $\quad \ell_K \triangleq \max \mathcal{X}$

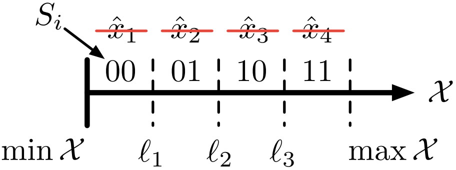
X_i MSi's channel capacity
$\mathcal{X}$ support set for r.v. X_i
S_i MSi's message to BS

Outline

Minimum Distortion Scalar Quantizers

Obs: Not reproducing local state ⇒ reconstruction points not needed

HomSQ: Both users have identical quantizer decision boundaries



- Distortion is a function of $\boldsymbol{\ell}$
- Select $\boldsymbol{\ell}$ to minimize $D(\boldsymbol{\ell})$

Theorem

If $\boldsymbol{\ell}$ is an optimal HomSQ then there exists $\boldsymbol{\mu} \geq 0$ such that

$$f_X(\ell_k) \int_{\ell_{k-1}}^{\ell_{k+1}} (\ell_k - x) f_X(x)\,\mathrm{d}x - \mu_k + \mu_{k+1} = 0$$

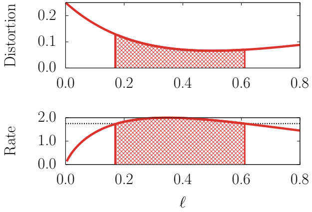
$$\mu_k(\ell_{k-1} - \ell_k) = 0.$$

Entropy Constrained Scalar Quantizers

- Rate is also a function of $\boldsymbol{\ell}$
- Select $\boldsymbol{\ell}$ to minimize $D(\boldsymbol{\ell})$ w/ an upper-limit R_0 on rate

$$p_k \triangleq \mathbb{P}(S_i = k) = F_X(\ell_k) - F_X(\ell_{k-1})$$

$$R_{HomSQ}(\boldsymbol{\ell}) = H(S_1) + H(S_2) = 2H(s)$$



Problem is non-convex

Theorem

If $\boldsymbol{\ell}$ is an optimal ECSQ, then there exists $\boldsymbol{\mu} \geq 0$ and $\mu_R \geq 0$ such that

$$f_X(\ell_k)\left(\int_{\ell_{k-1}}^{\ell_{k+1}} (\ell_k - x) f_X(x)\,\mathrm{d}x + 2\mu_R \log_2\left(\frac{p_{k+1}}{p_k}\right)\right) - \mu_k + \mu_{k+1} = 0$$

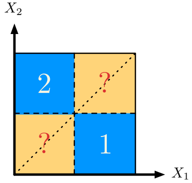
$$\mu_k(\ell_{k-1} - \ell_k) = 0 \text{ and } \mu_R(R_{HomSQ}(\boldsymbol{\ell}) - R_0) = 0.$$

Outline

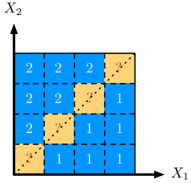
From HomSQs to HetSQs

Some Insights

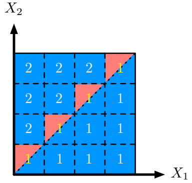
HomSQ 1



HomSQ 2



Deterministic HomSQ



- Obvious for $S_1 \neq S_2$
- Flip a coin for $S_1 = S_2$
- Distortion only along diagonal

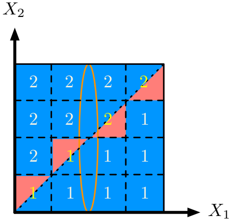
- arg max not symmetric
- Distortion *is*
- Same distortion for a fixed mapping along diagonal

From HomSQ to HetSQ

Rate Reduction

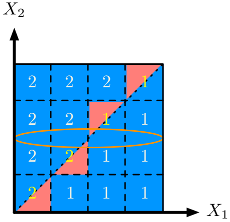
Obs: All mappings have the same distortion; some have better total rate

Mapping One



$R^{(1)}_{HetSQ} \leq R^{(1)}_{HomSQ}$

Mapping Two



$R^{(2)}_{HetSQ} \leq R^{(2)}_{HomSQ}$

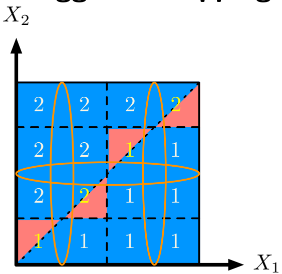
Follows from subadditivity of $t \log t$

Theorem

For an optimal HomSQ ℓ^* that achieves a distortion $D(\ell^*)$, there exists a HetSQ that achieves the same distortion but at a lower rate.

Staggered HetSQ

Staggered Mapping



$$R^{(1)}_{HetSQ} \leq R^{(1)}_{HomSQ}$$
$$R^{(2)}_{HetSQ} \leq R^{(2)}_{HomSQ}$$

Theorem

For a HetSQ, if there exists an quantization interval for a user that is completely contained in the quantization interval for another user, then the quantizer is not optimal.

Design of HetSQ

1: Select the total # of bins K

2: Design optimal HomSQ boundaries ℓ_{HomSQ}

3: Assign $\ell^{(k)}_{HomSQ}$ to MS1 if k odd; else, MS2

Outline

1. Introduction
2. Problem Model
3. Optimal Scalar Quantizer Design
   - Homogeneous Scalar Quantizers
   - Heterogeneous Scalar Quantizers
4. **Results**
5. Conclusions

Example 1

Uniform(a, b) Channel Capacity

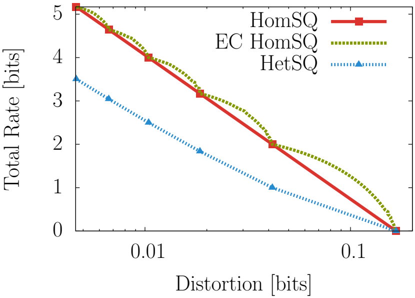
For $k = 1, \ldots, K-1$, the optimal quantizer is given as

$$\ell_k^* = \frac{aK + (b-a)k}{K}, \; \mu_k^* = 0$$

HetSQ—*No free lunch*; consider

- $K = 3$: 42.1% fewer bits
  - MS2 scheduled w.p. 0.556
  - $R_{HetSQ}^{(2)} = R_{HetSQ}^{(1)}$
- $K = 4$, 37.5% fewer bits
  - MS2 scheduled w.p. 0.500
  - $R_{HetSQ}^{(2)} = 0.67 R_{HetSQ}^{(1)}$

Uniform$(0,1)$ & $K = 1, \ldots, 6$



Example 2

Exponential (Exp(λ)) Channel Capacity

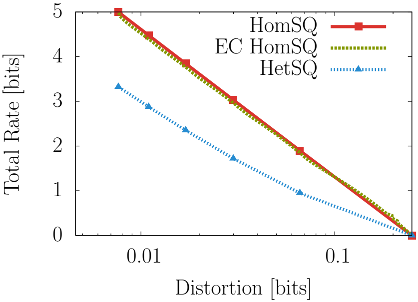
Let $w_k = \lambda \ell_k$; then the optimal quantizer is given as

$$w_k^* = \frac{-e^{-w_{k-1}^*}(1 + w_{k-1}^*) + e^{-w_{k+1}^*}(1 + w_{k+1}^*)}{\left(e^{-w_{k+1}^*} - e^{-w_{k-1}^*}\right)}$$

HetSQ—*No free lunch*; consider

- $K = 3$: 43.3% fewer bits
  - MS2 scheduled w.p. 0.560
  - $R_{HetSQ}^{(2)} \approx 0.73 R_{HetSQ}^{(1)}$
- $K = 4$: 38.9% fewer bits
  - MS2 scheduled w.p. 0.534
  - $R_{HetSQ}^{(2)} \approx 0.67 R_{HetSQ}^{(1)}$

Exp(2) & $K = 1, \ldots, 6$



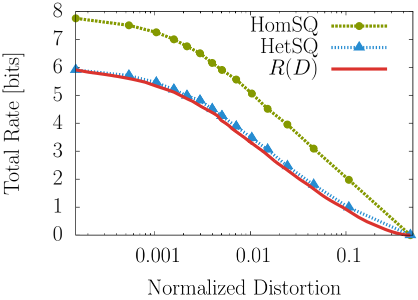
Example 3: LTE CQI

Discrete Uniform Channel Capacity

Q: Can we compare HomSQ and HetSQ to a fundamental limit?
A: R-D function computed from Berger-Tung bound
Berger-Tung inner & outer bounds coincide for independent sources

- 16 CQI levels in LTE [7]
- $D = 0$ for $R_1 + R_2 \leq 8$ bits
- HetSQ within 0.124 bits
- HomSQ within 1.622 bits

SQ w/ SW coding is optimal for recovery of discrete sources (Servetto 2005) [4]



Produced w/ help from Gwanmo Ku & Jie Ren

Outline

Review of Contributions

1. Distortion optimal HomSQs
2. Entropy-constrained distortion optimal HomSQs
3. Simple HetSQs achieve same distortion w/ lower rate as best HomSQs
4. HetSQ for discrete uniform distribution is close to fundamental limit

Future Work

- Fundamental limit for continous sources
- Generalize 2-user to N-user
- Consider VQs for multiple subbands
- Investigate low-rate performance as $N \to \infty$

Acknowledgments

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