An equivalence between network coding and index coding
—A paper by Effros, Rouayheb, Langberg [1]

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Outline

1. Network coding (NC) intro
2. Index coding (IC) intro
3. Mapping network coding to index coding
4. Equivalence proofs
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Network coding (NC) model

- Directed acyclic network $G = (V, E)$
- Source $s$ with rate $R_s$, edge $e$ with capacity $C_e$, sink node $t$ requires $\beta(t)$
- Independent sources
- Variables on out-going edges are function of variables on in-coming edges
- Goal: transmit information from source to sink via network codes
Any network has associated multiple unicast network

In [2], it is shown that for any multi-multicast network, there exists an associated multiple unicast network such that one is solvable if and only if the other is solvable.

- **Construction**
  - If a source message $a$ is transmitted by more than one sources, add a new super node to transmit $a$ to those sources.
Any network has associated multiple unicast network

- Construction (cont.)
  - If a source message $b$ is required by more than one sink nodes, add 5 new nodes to let $b$ is required by only one.
  - Iterations may be needed for the cases where more than 2 sinks requiring same message.
Multi-multicast to multi-unicast

Thus, W.O.L.G, we can only consider the multi-unicast networks.

- Source with index $k$ has rate $R_k$, edge $e$ with capacity $C_e$, sink node with index $k$ requires $Y_k$
- Independent sources
- Variables on out-going edges are function of variables on in-coming edges
- Goal: $K$ source/sink pairs wish to communicate
Achievable rate tuple in multi-unicast NC

Rate tuple $R = (R_1, \ldots, R_K)$ is achievable with block length $n$ (or $(R, n)$-feasible) if $\exists \{S_k\}, \{X_e\}$:

- **Rate**: Source $Y_k$ independent and uniform over $[2^{R_k n}]$
- **Edge capacity**: Edge $X_e$ with support $[2^{C_e n}]$
- **Functionality**: For edge $e$, is function of incoming edges, $X_e = f_e(X_{e1}, X_{e2}, \ldots)$
- **Decoding**: Sink $t_k$, requires $S_k$
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Index coding (IC) motivation

- A set of packets need to be delivered to a set of users
- Users may have some information already (side information)
- Broadcast channel is used by the server
- Goal: minimum transmission times

Users have:
- User 1: $M_2, M_4$
- User 2: $M_1, M_3$
- User 3: $M_2, M_4$
- User 4: $M_1$

Server has:
- $M_1, M_2, M_3, M_4$
Index coding motivation

- Traditional way: send $M_1, M_2, M_3, M_4$ respectively
- Coding: send $M_2 + M_3, M_1 + M_4$ respectively
- Side information is utilized
- Transmission times are reduced

Want: $M_1$
Has: $M_2, M_4$

Want: $M_2$
Has: $M_1, M_3$

Want: $M_3$
Has: $M_2, M_4$

Want: $M_4$
Has: $M_1$
Index coding model

The above example can be represented as a typical index coding model:

- Sources, sinks with requirements and side information, broadcasting node with coding
- Broadcast link has capacity $C_B$, others unlimited
Achievable rate in IC

Rate tuple $R = (R_1, \ldots, R_K)$ is achievable with block length $n$ (or $(R, n)$-feasible) if $\exists \{M_k\}, \{X_B\}$:

- **Rate**: Source $M_k$ independent and uniform over $[2^{R_k n}]$
  - **Edge capacity**: Edge $X_B$ with support $[2^{C_B n}]$
  - **Functionality**: For edge $B$, is function of incoming edges, $X_B = f_B(M_1, M_2, \ldots)$
  - **Decoding**: Sink $t_k$, requires $M_k$
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Reduce NC to IC

Instances in IC:

- Sources: NC sources and NC edges
- Sinks: NC sinks, NC edges and a special sink representing all edges are function of all NC sources
- For NC edge $e$, sink $t_e$ in IC wants $X_e$ and has as side information all IC sources incoming to $e$ in NC

IC encodes topology of NC in its sinks!
Reduce NC to IC

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Theorem

For any NC with rate tuple $R$, one can construct an IC with rate $R'$ such that for any block length $n$: NC is $(R, n)$-feasible iff IC is $(R', n)$-feasible.

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**Proof: NC→IC**

Basic idea: simulate NC solution in IC decoding.

- Each edge $e$ in NC has a global function $F_e(S_1, \ldots S_K) = X_e$
- Recall $X_e$ has support $[2^{C_e n}]$, $C_B = \sum C_e$ and $X_B$ has support $[2^{C_B n}]$
- $X_B$ can be divided into chunks to associated with $e$
- Let $X_B(e) = S'_e + F_e(S'_1, \ldots, S'_K)$

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Proof: NC $\rightarrow$ IC (cont.)

- **Decoding:**
  - consider $t'_e$: wants $S'_e$ and has $\{S'_a\}$ for edges $a \in \ln(e)$, also receives $X_B$
  - for each $a$, compute $X_B(a) - S'_a = F_a(S'_1, \ldots, S'_K)$
  - use local function $f_e$ to compute $f_e(F_a_1(S'_1, \ldots, S'_K) \ldots) = F_e(S'_1, \ldots, S'_K)$
  - Then decode $S'_e = X_B(e) - F_e(S'_1, \ldots, S'_K)$
  - Same process for other sinks

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Proof: IC → NC

- Encoding function $f_B(\{S'_k\}, \{S'_e\})$
- $t_{all}$ requires all $S'_E$ decodable given any $S'_K$ and $X_B$
- Crucial property: for any $S'_K$, $\exists S'_E$ such that $X_B = f_B(S'_K, S'_E) = \sigma$
- To define NC we need $X_B = \sigma$

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Proof: IC→NC (cont.)

- Consider edge \( e \) in NC, we need local encoding function
  \( f_e(X_{a1}, \ldots) = X_e \)
- We have decoding function for \( t'_e \) in IC: 
  \[ g'_e(S'_{a1}, \ldots, X_B) = S'_e \]
- Let 
  \[ f_e(X_{a1}, \ldots) = g'_e(X_{a1}, \ldots, \sigma) \]
- Similarly for sink \( k \) in NC, we need decoding function
  \( g_k(X_{a1}, \ldots) = S_k \)
- We have \( g'_k(S_{a1}, \ldots f_B) = S'_k \), and then let
  \[ g_k(X_{a1}, \ldots) = g'_k(X_{a1}, \ldots, \sigma) \]
- Let \( S_I = S'_I, X_e = S'_e \), can show the correctness of enc. (dec.)
Proof: IC→NC (cont.)

- Encoding:

\[ X_e = f_e(X_{a1}, \ldots) \]
\[ = g'_{e}(X_{a1}, \ldots, \sigma) \]
\[ = g_{e}(X_{a1}, \ldots, f_{B}(S'_{I}, S'_{E})) \]
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Proof: IC $\rightarrow$ NC (cont.)

- Decoding:

\[ \hat{S}_i = g_i(X_{a1}, \ldots) \]
\[ = g'_i(X_{a1}, \ldots, \sigma) \]
\[ = g'_i(S'_{a1}, \ldots, f_B(S'_I, S'_E)) \]
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