$U_{2,4}$–The only forbidden minor for binary matroids

Congduan Li

ASPITRG & MANL
Drexel University

congduan.li@gmail.com

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Outline

1. Background of Matroid Theory
   - Definitions of matroids
   - Representable matroids
2. The forbidden minor for binary matroids
   - Binary matroids
   - Theorem statement
   - $U_{2,4}$ is a forbidden minor for binary matroids
   - $U_{2,4}$ is the only forbidden minor for binary matroids

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Def: Independent sets

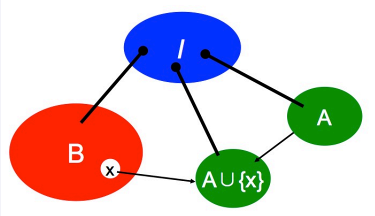
Definition

A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E having the following three properties:

1. $\phi \in \mathcal{I}$;
2. Hereditary:
   If $I \in \mathcal{I}$ *and* $I' \subseteq I$, *then* $I' \in \mathcal{I}$;
3. Augmentation:
   If I_1 *and* I_2 *are in* $\mathcal{I}$ *and* $|I_1| < |I_2|$, *then there is an element* e *of* $I_2 - I_1$ *such that* $I_1 \cup e \in \mathcal{I}$.

Demo

Augmentation



Def: Base/Basis

A collection of subsets $\mathcal{B} \subseteq 2^E$ of a ground set E are the bases of a matroid if and only if:

- $\mathcal{B}$ is non-empty;
- base exchange: If $B_1, B_2 \in \mathcal{B}$, for any $x \in (B_1 - B_2)$, there is an element $y \in (B_2 - B_1)$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

Consider

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5}

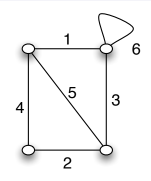
Def: Circuits

The circuits $\mathcal{C}(M)$ of a matroid are the minimal dependent sets.

Property

- If any one element is deleted from a circuit, it will become an independent set.
- Independent sets $\mathcal{I}(M)$ does not contain any element in $\mathcal{C}(M)$.

Consider the graph



Circuits: {1,2,3,4}, {1,3,5}, {2,4,5}, {6}

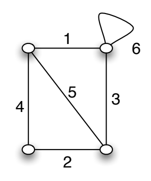
- Size 1 circuit is a loop
- Size 2 circuit contains parallel elements
- Simple matroids: no loops or parallel elements

Def: Circuits

A collection of sets $\mathcal{C}$ is the collection of circuits of a matroid if and only if:

- $\phi \notin \mathcal{C}$
- if $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$ then $C_1 = C_2$
- if $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there is some $C_3 \in C$ with $C_3 \subseteq (C_1 \cup C_2) - e$.

Consider the graph



Circuits: {1,2,3,4}, {1,3,5}, {2,4,5}, {6}

Def: Rank function

For any base of a matroid, $r(B) = r(M)$.
Formally, a rank function is $r : 2^E \to \mathbb{N} \cup \{0\}$, for which

- For $X \subseteq E, 0 \leq r(X) \leq |X|$
- If $X \subseteq Y \subseteq E, r(X) \leq r(Y)$
- $\forall X, Y \subseteq E, r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$

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Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5};
Rank: $r(M) = r(B) = 3$

Closure

Given a rank function of a matroid, the closure operation $\mathrm{cl} : 2^{\mathrm{E}} \to 2^{\mathrm{E}}$ is defined as

$$\mathrm{cl}\ X = \{x \in E | r(X \cup x) = r(X)\}.$$

Closure is a span of a subspace.
$\mathrm{cl}\ B = E(M)$;
Independent Set:
$\mathcal{I} = \{X \subseteq E | x \notin \mathrm{cl}(X - x) \forall x \in X\}$

A subset X of $E(M)$ is said to be a flat if $\mathrm{cl}\ \mathrm{X} = \mathrm{X}$.
$r(M) - 1$ flat is a hyperplane.
Consider

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cl $\{1,2,6\}=\{1,2,6\} \to$ flat, hyperplane
cl $\{1,3\}=\{1,3,5,6\}$

Closure

A function $\mathrm{cl} : 2^{\mathrm{E}} \rightarrow 2^{\mathrm{E}}$ is closure for a matroid $M = (E, \mathcal{I})$ iff

- If $X \subseteq E$ then $X \subseteq \mathrm{cl}X$
- If $X \subseteq Y \subseteq E$, then $\mathrm{cl}X \subseteq \mathrm{cl}Y$
- If $X \subseteq E$ then $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}X$
- If $X \subseteq E$ and $x \in E$ and $y \in \mathrm{cl}(X \cup x) - \mathrm{cl}(X)$ then $x \in \mathrm{cl}(X \cup y)$

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cl $\{1,2\}$=$\{1,2,6\}$ $\rightarrow$ flat

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Overview

Representable Matroids

Definition

A matroid M with ground set S of size $|S| = N$ and rank $r_M = r$ is representable over a field $\mathbb{F}$ if there exists a matrix $A \in \mathbb{F}^{r \times N}$ such that for each independent set $I \in \mathcal{I}$ the corresponding columns in A, viewed as vectors in $\mathbb{F}^r$, are linearly independent.

$$S_1, \dots, S_i, \dots, S_j, \dots, S_N \overset{Mapping}{\Longleftrightarrow} 1\ 2\ \dots\ i\ \dots\ j\ \dots\ N$$

$$r(S_i, S_j) = rank(i, j\text{-th columns}) \qquad \begin{array}{ccccc} & i & & j & \\ \Bigg[\ \dots & \big(\ \big) & \dots & \big(\ \big) & \dots \ \Bigg] \end{array}$$

- Representable matroids usually can be characterized by forbidden minors (cannot contain such minors).

Minors of Matroids

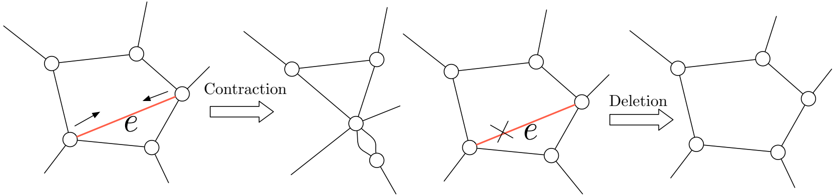
Definition

If M is a matroid on S and $T \subseteq S$, a matroid M' on T is called a *minor* of M if M' is obtained by any combination of deletion ($\backslash$) and contraction ($/$) of M.

Contraction and Deletion

Let M/T denote the matroid obtained by contraction of M on $T \subset S$, and let $M \setminus T$ denote the matroid obtained by deletion from M of $T \subset S$. Then, $\forall X \subseteq S - T$

$$\begin{aligned} r_{M/T}(X) &= r_M(X \cup T) - r_M(T) \\ r_{M \setminus T}(X) &= r_M(X) \end{aligned} \tag{1}$$



Overview

Binary Matroids

- $M = M(A)$, where A is a binary matrix.
- Unique representation under equivalence: $A = [I_r | D]$, where I_r is identity matrix representing a basis B, $D_{e,f} = 1$ or 0, if $B - e + f$ is or is not a basis.
- Example: selected basis $\{a, b, c\}$, other bases $\{b, c, d\}, \{a, b, d\}, \{b, c, e\}, \{a, c, e\}$

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

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Theorem

Theorem (Tutte 1958)

A matroid is binary if and only if it has no $U_{2,4}$-minor.

$U_{2,4}$: rank=2, for any 2 or more of the total 4 elements.

Proof points:

1. $U_{2,4}$ is not binary representable and so its extensions. Hence, $U_{2,4}$ is a forbidden minor;
2. $U_{2,4}$ is the only such forbidden minor: suppose exists another forbidden minor, equivalent to $U_{2,4}$

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Point 1: $U_{2,4}$ is a forbidden minor

- $U_{2,4}$ is not binary representable due to limit of numbers of binary vectors: $A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & ? \end{bmatrix} \end{array}$
- Extensions of a not-$\mathbb{F}_q$-representable matroid will not be $\mathbb{F}_q$ representable: Suppose a matroid is $\mathbb{F}_q$-representable by a matrix, deletion/contraction on associated columns in the matrix will give representation for its minors, obtained by some deletion/contraction.
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Point 2.0: Suppose existence of another forbidden minor

- Forbidden minors: simple, not binary, but minors are binary.
- Suppose exists another forbidden minor M.
- M has no $|E(M)|-1$- or $|E(M)|-2$-element hyperplane;
  - if has $|E(M)|-1$-element hyperplane, representation for M by adding column to the representation matrix $[I_{r-1}|D]$ of the hyperplane:
    $\begin{bmatrix} I_{r-1} & \mathbb{O} & D \\ 0 & 1 & \mathbb{O} \end{bmatrix}$
  - if has $|E(M)|-2$-element hyperplane, representation will be:
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Point 2.1: Representation and M'

- For any distinct $x, y \in E(M), r(M \setminus \{x, y\}) = r(M)$.
- Suppose $[I_r|D] = M \setminus \{x, y\}$ over binary. $[I_r|D|v_x] = M \setminus \{y\}$ and $[I_r|D|v_y] = M \setminus \{x\}$. Let $M' = [I_r|D|v_x|v_y]$.
- $M \setminus x = M' \setminus x$ and $M \setminus y = M' \setminus y$, but $M \neq M'$.
- $\exists$ a set s.t. independent in one but dependent in the other.
- Let Z be such a minimal set. $Z \in \mathcal{I}(M_I), Z \in \mathcal{C}(M_C), \{M_I, M_C\} = \{M, M'\}$. (if $Z \notin \mathcal{C}(M_C)$, $\exists$ some elements to delete and keep the dependence, contradicting with the minimality)
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- Want to show: $Z = \{x, y\}$ & $r_{M_I}(E) = 2$
- Consequences of: if $Z \subseteq J, J \in \mathcal{I}(M_I)$, then $J = \{x, y\}$.
- Suppose $G = J - \{x, y\} \neq \emptyset$. $G \in \mathcal{I}(M_I \setminus \{x, y\})$ and $G \in \mathcal{I}(M_C \setminus \{x, y\})$.
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- Consider a basis B for $N_I \setminus \{x, y\}$, which equals $N_C \setminus \{x, y\}$. B is either a basis of both N_I, N_C or of neither. (reason: N_I, N_C same rank)
- If B is a basis of neither, $r_{N_I}(E \setminus G \setminus \{x, y\}) < r_{N_I}(E \setminus G)$, contradicting with the fact that no $|E(M)| - 1$- or $|E(M)| - 2$-element hyperplane.
- If add back $r_{N_I}(G)$ to both sides, get $r_{M_I}(E \setminus \{x, y\}) < r_{N_I}(E)$

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Point 2.3: $M = U_{2,4}$

- B is a basis of both N_I, N_C, both are binary.
- $N_I \setminus x = N_C \setminus x = [I_{r'}|D'|v'_y]$ and $N_I \setminus y = N_C \setminus y = [I_{r'}|D'|v'_x]$, $[I_{r'}|D'|v'_x|v'_y] = N_I = N_C$ due to uniqueness. Contradiction.
- $\{x, y\} \subseteq Z \subseteq J = \{x, y\}$. A basis $J \supseteq Z$ is {x,y}, we have $r_{M_I}(E) = 2$.
- $r_{M_I}(E) = r_{M_C}(E) = r_{M_I}(E \setminus \{x, y\}) = r_{M_C}(E \setminus \{x, y\}) = 2$.
- Consider every possible x, y, M, either M_I, M_C, has rank 2 and $\{x, y\}$ is a basis, at least 4 elements (if ≤ 4, M is binary), is simple.
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