Solving Rota's Conjecture*

Jayant Apte
ASPITRG

Outline

• Background
• What is Rota's conjecture?
• Graph Minors and WQO theorem for graphs
• Graph minors Structure theorem
Outline

- Background
- What is Rota's conjecture?
- Graph Minors and WQO theorem for graphs
- Graph minors Structure theorem
Matroid

- A matroid consists of a pair $E, \mathcal{I}$ where $E$ is a finite set and $\mathcal{I}$ is a collection of subsets of $E$
- $E$ is called the ground set
- Subsets in $\mathcal{I}$ are called independent sets
- $(E, \mathcal{I})$ obey certain axioms:
  
  I1 $\emptyset \in \mathcal{I}$
  
  I2 Subsets of independent sets are independent
  
  I3 For each $X \subseteq E$, the maximal independent sets of $X$ have the same size
Cryptomorphic axiom systems

- \((E, \mathcal{B})\): where \(\mathcal{B}\) is collection of all maximal independent sets of \(E\)

- \((E, r)\): \(r : 2^E \rightarrow \mathbb{Z}_{\geq 0}\) is called the rank function

- These generalize the idea of bases of vector space and linear rank
Column matroid

- Let $A$ be a matrix over field $\mathbb{F}$
- Let $E$ be the set of column indices of $A$
- Let $\mathcal{I}$ be the collection of subsets of $E$ that index linearly independent sets of columns
- The column matroid of $A$, denoted as $M(A)$ is $(E, \mathcal{I})$
F-representability

- A matroid is called F-representable if it is column matroid of some matrix over F

- **Conjecture** The proportion of $n$ element matroids that are representable is vanishingly small as $n \to \infty$
Whitney's Problem(s)


Are following problems *decidable*?*

(P1) Is the given matroid representable over *any* field?

YES!


(P2) Is the given matroid representable over *given finite field*?

Rota's Conjecture!

*A decision problem is decidable if there exists a finite terminating algorithm to solve it*
Whitney's Problem(s)


Are following problems *decidable*? 

(P1) Is the given matroid representable over *any* field?

(P2) Is the given matroid representable over *given finite field*?

*A decision problem is decidable if there exists a finite terminating algorithm to solve it*
Whitney's Problem(s)


Are following problems *decidable*? 

(P1) Is the given matroid representable over *any* field?

   YES!


(P2) Is the given matroid representable over *given finite field*?

   Rota's Conjecture!

* A decision problem is decidable if there exists a finite terminating algorithm to solve it.
Matroid Duality

- *Dual* of a matroid $M$ is a matroid denoted as $M^*$ whose bases are complements of those of $M$

- **Theorem** A matroid $M$ is $\mathbb{F}$-representable if and only if $M^*$ is $\mathbb{F}$-representable
Deletion and Contraction

- Let $C$ and $D$ be sets of elements in a matroid $M = (E, \mathcal{I})$

- The matroid obtained by deleting $D$ is defined as $(E - D, \{I \subseteq E - D : I \in \mathcal{I}\})$

- *Contraction* is dual operation of deletion

- $(M/C) = (M^\star \setminus C)^\star$
Minors

- Minors of a matroid $M$ are matroids of type $M \setminus D/C$
- A minor is called *proper* if $D \cup C$ is nonempty
- The class of $\mathbb{F}$-representable matroids is closed under deletion and duality
- If a matroid is $\mathbb{F}$-representable, so are all its minors
- Hence the class of $\mathbb{F}$-representable matroids is closed under taking minors
Outline

● Background

● What is Rota's conjecture?

● Graph Minors and WQO theorem for graphs

● Graph minors Structure theorem
Excluded Minors

- Naturally we start look for matroids that are not $\mathbb{F}$-representable but all their proper minors are.

- We call these excluded minors.

- Conjecture (Rota): For each finite field $\mathbb{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.
Excluded Minors

• Naturally we start look for matroids that are not $\mathcal{F}$-representable but all their proper minors are

• We call these excluded minors

• Conjecture (Rota): For each finite field $\mathcal{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\mathcal{F}$-representable matroids
Excluded Minors

- Naturally we start look for matroids that are not $\mathbb{F}$-representable but all their proper minors are.

- We call these excluded minors.

- **Conjecture (Rota):** For each finite field $\mathbb{F}$, there are, up to isomorphism, **only finitely many** excluded minors for the class of $\mathbb{F}$-representable matroids.

- (P2) is decidable if this is true.
Outline

• Background
• What is Rota's conjecture?
• Graph Minors and WQO theorem for graphs
• Graph minors Structure theorem
Graph Minors

- A *minor* of a graph $G$ is a graph that is obtained from a subgraph of $G$ by *contracting* some edges.
Characterization of planar graphs

- Planar graphs are graphs that can be embedded in a plane i.e. they can be drawn in such a way that no edges cross each other

- Planar graphs are closed under taking minors

- Kuratowski’s Theorem: A graph is not planar iff it has a minor isomorphic to $K_{3,3}$ or $K_5$
  
  \begin{footnotesize}
  \text{(K. Kuratowski, Sur le problème des courbes gauches en topologie, Fund. Math. 15 (1930), 271283.)}
  \end{footnotesize}
Generalized Kuratowski Theorem

- Robertson and Seymour generalized Kuratowski’s theorem from plane to arbitrary surfaces.

Theorem: For any given surface, there are only finitely many excluded minors for the class of graphs that embed in the surface.

A partial order on set of non-isomorphic graphs

- Let $\mathcal{G}$ be set of all non-isomorphic finite undirected graphs

- If $G$ can be obtained from $H$ by taking minor, we say $G \leq H$

- '$\leq$' is a partial order on $\mathcal{G}$ since it is:

  P1 Reflexive
  Every graph is minor of itself

  P2 Transitive
  Every minor of $G$ is itself a minor of $G$

  P3 Antisymmetric
  If $G$ and $H$ are minors of each other, then they must be isomorphic
A quasi-order on set of all graphs

- Let $\mathcal{G}$ be set of all finite undirected graphs
- If $G$ can be obtained from $H$ by taking minor, we say $G \preceq H$
- $\preceq$ is quasi-order on $\mathcal{G}$ since it is:
  - P1 Reflexive
    Every graph is minor of itself
  - P2 Transitive
    Every minor of $G$ is itself a minor of $G$
Well-founded quasi order

- A quasi order $R$ on $X$ is Well-founded if

WQO1 There are no infinite descending chains
  (infinite sequences of type $x_1 > x_2 > x_3, \ldots, x_i \in X$)

WQO2 There are no infinite antichains
  (infinite subsets with pairwise incomparable elements
  $A \subseteq X$ s.t. neither $x_i \leq x_j$ nor $x_j \leq x_i$, $\forall x_i, x_j \in A$)

- Set of all graphs:

WQO1 is true since starting with a finite graph and deleting
  /contracting edges must end at empty graph

WQO2 the non-trivial part
Why is WQO so important?

A2 In any infinite set of graphs, there exists a pair $x, y$ that is comparable under minor relation $\leq$.

A1 No infinite antichains.

A3 Let $S$ be an infinite subset of $\mathcal{G}$, and $M$ be the minor minimal subset of $S$ i.e. $\forall x \in M$ if $y \leq x \Rightarrow y \notin S$; then $M$ is finite.
Why is WQO so important?

**A1** No infinite antichains

**A2** In any infinite set of graphs, there exists a pair $x, y$ that is comparable under the minor relation $\leq$.

**A3** Let $S$ be an infinite subset of $G$ and $M$ be the minor minimal subset of $S$ i.e. $\forall x \in M$ if $y \leq x \Rightarrow y \notin S$, then $M$ is finite.

$M$ is an antichain for any $S$. If not, there exists $x \in M$ such that $y \leq x$ and $y \in M$, thus contradicting minor minimality of $x$. 
Why is WQO so important?

In any infinite set of graphs \( \exists \) a pair \( x, y \) that is comparable under minor relation \( \leq \)

**A1**

No infinite antichains

**A3**

Let \( S \) be an infinite subset of \( \mathcal{G} \) and \( M \) be the minor minimal subset of \( S \) i.e. \( \forall x \in M \) if \( y \leq x \) \( \Rightarrow y \notin S \), Then \( M \) is finite

Obvious
Why is WQO so important?

A1 No infinite antichains

A2 In any infinite set of graphs \( \exists \) a pair \( x, y \) that is comparable under minor relation \( \leq \)

Upwards: Every element in \( S \setminus M \) is comparable to at least one element of \( M \)

Downwards: If \( M \) was to be infinite, \( y \in M \) s.t. \( y \leq x \), i.e. \( x \in M \) has a minor that is also in \( M \) \( \Rightarrow \iff \)

A3 Let \( S \) be an infinite subset of \( \mathcal{G} \) and \( M \) be the minor minimal subset of \( S \) i.e. \( \forall x \in M \) if \( y \leq x \) \( \Rightarrow y \notin S \), Then \( M \) is finite
Why is WQO so important?

A1 No infinite antichains

A2 In any infinite set of graphs, \( \exists \) a pair \( x, y \) that is comparable under minor relation \( \leq \)

A3 Let \( S \) be an infinite subset of \( \mathcal{G} \) and \( M \) be the minor minimal subset of \( S \) i.e. \( \forall x \in M \) if \( y \leq x \Rightarrow y \notin S \), Then \( M \) is finite
Why is WQO so important?

A1 No infinite antichains

A2 In any infinite set of graphs \( \exists \) a pair \( x, y \) that is comparable under minor relation \( \leq \)

A3 Let \( S \) be an infinite subset of \( \mathcal{G} \) and \( M \) be the minor minimal subset of \( S \) i.e. \( \forall x \in M \) if \( y \leq x \Rightarrow y \notin S \), Then \( M \) is finite
WQO and minor closed families of graphs

$F$: an (infinite) minor closed family

$F^c$

$M$: Minor minimal subset of $F^c$
WQO and minor closed families of graphs

$F$: an (infinite) minor closed family

$F^c$

Forbidden minors of $F$

This is finite!
A further generalization: WQO Theorem

**Theorem:** Each minor-closed class of graphs has only finitely many excluded minors


Alternatively,

- In each infinite set of graphs, there are two graphs, one is isomorphic to a minor of the other
- There are countably many distinct minor closed classes of graphs
Excluded minors for planarity

\[ K_5 \]

\[ K_{3,3} \]
Example: Peterson graph
Example: Peterson graph  Not Planar
Other minor closed families with forbidden minor characterizations

Apex Graphs:
Graphs that can be made planar by removal of a single vertex
Finite (unknown) excluded minors
Other minor closed families with forbidden minor characterizations

Graph Unions* of Cactus graphs:
Graph unions of graphs in which any two simple cycles contain at most 1 common vertex
One Excluded minor: Dimond graph

* A binary operation corresponding to disjoint union of vertex sets and edge sets
Other minor closed families with forbidden minor characterizations

**Pseudoforests**: Graphs having at most one cycle in every connected component

**Two Excluded minors:**
1) Dimond graph 2) Butterfly graph

* A sub-class of unions of cactus graphs
Outline

- Background
- What is Rota's conjecture?
- Graph Minors and WQO theorem for graphs
- Graph minors Structure theorem
Graph Minors Structure Theorem

- The set of excluded minors $M$ form an antichain under the well quasi order
- Let $F = (H_1, H_2, H_3, \ldots)$
- A graph is said to be $H_1$-free if has no minor isomorphic to $H_1$
- $(H_2, H_3, \ldots)$ must be $H_1$-free (having no minor isomorphic to $H_1$)
- Let $EX(H_1)$ be the set of all graphs not having a minor isomorphic to $H_1$
- If $H_1$ does not embed into a surface $S$, graphs that embed into surface $S$ must be in $EX(H_1)$. Call this set $P$.
- Graph minors structure theorem provides a means to construct graphs in $EX(H_1)$ from $P$
Part-2 Outline

- Graphs to matroids: Forbidden minor characterization of graphic matroids
- Matroid WQO theorem
- Projectively Inequivalent Representations
- Bifurcation
- Representability under circuit hyperplane relaxation
Note: forbidden minor characterization perfect graphs

• Minors (in the sense we discussed so far) yield a relation on set of all graphs

• So do induced subgraphs

• Perfect graphs are characterized by forbidden induced minors

• Forbidden minors*: Odd holes (simple cycles of length not less than 5) and their complements

Chudnovsky, Maria; Cornuéjols, Gérard; Liu, Xinming; Seymour, Paul; Vušković Kristina (2005). "Recognizing Berge graphs". Combinatorica 25 (2): 143–186
Note: WQO for Digraphs?

• What relation one should use?
  – Edge contraction is absurd
Note: WQO for Digraphs?

- What relation one should use?
  - Edge contraction is absurd
Note: WQO for Digraphs?

- What relation one should use?
  - Edge contraction is absurd
Note: WQO for Digraphs?

- What relation one should use?
  - Edge contraction is absurd
Note: WQO for Digraphs?

- What relation one should use?
  - Edge contraction is absurd
  - WQO fails
Note: WQO for Digraphs?

• How to deal with this problem?
• Consider a subclass: Semi-complete Tournaments*
• Various minor relations
  – Immersion
  – Strongly connected subgraph contraction

*A directed graph obtained from a complete undirected graph by orienting edges
Note: WQO for Digraphs?

- How to deal with this problem?
- Consider a subclass: Semi-complete Tournaments
- Various minor relations
  - Immersion: forms WQO on semi-complete tournaments*
  - Strongly connected subgraph contraction

Note: WQO for Digraphs?

- How to deal with this problem?
- Consider a subclass: Semi-complete Tournaments
- Various minor relations
  - Immersion: forms a WQO on semi-complete tournaments*
  - Strongly connected subgraph contraction: forms a WQO on semi-complete tournaments*

Part-2 Outline

- Graphs to matroids: Forbidden minor characterization of graphic matroids
- Matroid WQO theorem
- Projectively Inequivalent Representations
- Bifurcation
- Representability under circuit hyperplane relaxation
Graphic Matroids

\[
\begin{align*}
\text{a} & \quad \text{b} & \quad \text{c} & \quad \text{d} & \quad \text{e} \\
1 & \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \end{bmatrix} \\
2 & \quad \begin{bmatrix} -1 & 0 & 1 & -1 & 0 \end{bmatrix} \\
3 & \quad \begin{bmatrix} 0 & 1 & -1 & 0 & -1 \end{bmatrix} \\
4 & \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\end{align*}
\]
Graphic Matroids

\begin{align*}
\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}
\end{align*}

\begin{align*}
\begin{array}{ccccccc}
& a & b & c & d & e \\
1 & 1 & -1 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 1 & -1 & 0 & \\
3 & 0 & 1 & -1 & 0 & -1 & \\
4 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}
\end{align*}
Graphic Matroids

This is both binary and ternary representation.

\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 0 \\
0 & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
Graphic Matroids

Theorem (Seymour) A matroid is representable over all fields iff it is representable over both $GF(2)$ and $GF(3)$

Theorem  Graphic matroids are representable over all fields
Characterization of graphic matroids

Graphic matroids are a minor closed family

**Theorem (Whitney):** A graph $G$ is planar iff $M(G)^*$ is graphic
Characterization of graphic matroids

Graphic matroids are a minor closed family
Theorem (Whitney): A graph $G$ is planar iff $M(G)^*$ is graphic

Theorem: A matroid is graphic if it is binary

Theorem: A matroid is graphic if it is ternary
Characterization of graphic matroids

Graphic matroids are a minor closed family

**Theorem (Whitney):** A graph $G$ is planar iff $M(G)^*$ is graphic

Exclude $M(K_{3,3})^*$, $M(K_5)^*$!

**Theorem:** A matroid is graphic if it is binary

Exclude $U_{2,4}$!

**Theorem:** A matroid is graphic if it is ternary

Exclude $F_7$ and $F_7^*$!

*Excluding $U_{2,4}$ takes care of $U_{2,5}$ and $U_{3,5}$*
Characterization of graphic matroids

Graphic matroids are a minor closed family

**Theorem (Whitney):** A graph $G$ is planar iff $M(G)^*$ is graphic

**Theorem (Tutte):** The excluded minors for the class of graphic matroids are $U_{2,4}, F_7, (F_7)^*, M(K_{3,3})^*, M(K_5)^*$

Part-2 Outline

- Graphs to matroids: Forbidden minor characterization of graphic matroids
- **Matroid WQO theorem**
- Projectively Inequivalent Representations
- Bifurcation
- Representability under circuit hyperplane relaxation
Matroid WQO and rota's conjecture

- A WQO theorem for set of all finite matroids would imply Rota's conjecture is True
Matroid WQO and rota's conjecture

- A WQO theorem for set of all finite matroids would imply Rota's conjecture is True
- Unfortunately WQO theorem doesn't hold for all matroids
Matroid WQO

**Theorem (Matroid WQO)** For each finite field $\mathbb{F}$ and each minor closed class of $\mathbb{F}$-representable matroids, there are only finitely many $\mathbb{F}$-representable excluded minors.
Matroid WQO

Theorem (Matroid WQO) For each finite field $\mathbb{F}$ and each minor closed class of $\mathbb{F}$-representable matroids, there are only finitely many $\mathbb{F}$-representable excluded minors

Matroid WQO does not say anything about Rota’s conjecture
Part-2 Outline

- Graphs to matroids: Forbidden minor characterization of graphic matroids
- Matroid WQO theorem
- **Projectively Inequivalent Representations**
- Bifurcation
- Representability under circuit hyperplane relaxation
Projective equivalence of matroid representations

- Two matrices $A$ and $B$ are said to be *projectively equivalent* if $A$ can be obtained from $B$ by:
  - Elementary row operations
  - Column scaling
Projective equivalence of matroid representations

Projectively inequivalent representations of 3-whirl over GF(5)

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 2 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 3 \\
\end{bmatrix}
\]