Coding for Computing

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Outline

- Background and definitions
- Main Results
- Examples and Analysis
- Proofs
Background and definitions

● Problem Setup
● Graph Entropy
● Conditional Graph Entropy
● Characteristic Graph
● Wyner-Ziv
Problem Setup
Problem Setup

- Encode by block
  Transmits after observing the whole block of X
- Decode by block
  $f(X,Y)$ must be determined for a block of $(X,Y)$
- Vanishing block-error probability
  $$p(Z^N \neq f(X^N,Y^N)) \leq \varepsilon$$
- Problem’s rate
  $$L_f(X|Y) = \frac{R}{N}$$
Graph Entropy

Maximum independent sets of $G(V,E)$

$\Gamma(G) = \{\{1,3,5\}, \{1,3,6\}, \{1,4,6\}, \{2,5\}, \{2,6\}\}$
Graph Entropy

Define random viable $W$, $W \in \Gamma(G)$

$$p(w|x) = 0 \text{ if } x \text{ is not in } w$$

$$\sum_{w : x \in w} p(w|x) = 1$$
Graph Entropy:

\[ H_G(X) = \min_{p(w|x)} I(X; W) \]
Conditional Graph Entropy

Extend the definition of graph entropy:

Let \((X,Y)\) be a random pair and let a graph \(G\) be defined over the support set of \(X\)

\[
H_G(X|Y) = \min_{W \subseteq \{X\} \cup \{Y\}} \max_{W \subseteq \{X\} \cup \{Y\}} \sum_{W \subseteq \{X\} \cup \{Y\}} I(W;X|Y)
\]
Characteristic Graph

- $G(V,E)$ and $f$
- The vertex set is the support set of $X$
- The edge $(x,x')$ exists if there is a $y$ such that

$$p(x,y) > 0 \quad p(x',y) > 0$$

$$f(x,y) \neq f(x',y)$$
Outline

- Background and definitions
- Main Results
- Examples and Analysis
- Proofs
Main Results

- Naive bound
- Theorem 1
- Theorem 2
- Theorem 3
Main Results (Naive bounds)

**Inner bound**

\[ L_f(X|Y) \geq H(f(X, Y)|Y) \]

**Outer bound**

\[ L_f(X|Y) \leq \min \{ H(g(X)|Y) : g(X) \text{ and } Y \text{ determine } f \} \]

\[ H(f(X, Y)|g(X), Y) = 0 \]
Main Results (Naive bounds)

- **lower bound**: number of bits required when $P_x$ knows $Y$ in advance
- **upper bound**: using Slepian-Wolf theorem
- **Both bounds** are tight in some special cases but not in general
Main Results (Theorem 1)

Theorem 1 for every $X$, $Y$ and $f$

\[ L_f(X|Y) = H_G(X|Y) \]
Zero Error Worst Length Case

One-Way Complexity

\[ \hat{\mathcal{C}}_1(X|Y) = \lceil \log_\omega(G(X|Y)) \rceil \]

The one-way complexity is the chromatic number of the characteristic hypergraph of (X,Y)
Main Results (Theorem 2)

- An extension of Wyner-Ziv
- Allowing the distortion to depend on $Y$ as well as on $X$ and $Z$

$$R_D = \min_{p(u|x), g|E[d(x,y,g(u,y))]| \leq D, U \leftrightarrow X \leftrightarrow Y} I(X; U|Y)$$
Main Results (Theorem 2)

- Applicant in our problem

\[ d(x, y, z) = \begin{cases} 0, & \text{if } z = f(x, y) \\ 1, & \text{otherwise} \end{cases} \]

\[ d(X, Y, Z) = p(Z \neq f(X, Y)) \]

\[ d(X^N, Y^N, Z^N) = \frac{1}{N} \sum_{i=1}^{N} p(Z_i \neq f(X_i, Y_i)) \]
Main Results (Theorem 2)

Lemma: For every $X$, $Y$ and $f$

\[
L_f(X|Y) \geq R(0)
\]

\[
p(Z^N \neq f(X^N, Y^N)) \geq \frac{1}{N} \sum_{i=1}^{N} p(Z_i \neq f(X_i, Y_i))
\]

hence every rate achievable with vanishing block error is also achievable with zero distortion
Main Results (Theorem 2)

Theorem 2 For every $X, Y$ and $f$

$$R(0) = H_G(X|Y)$$
Main Results

Theorem 3 For every $X$, $Y$ and $f$

$$R_f^2(X|Y) = \{ (r_x, r_y) : r_x \geq I(V; X|UY) \text{ and } r_y \geq I(U; Y|X) \}$$

for some admissible $U$ and $V$
Proof of Theorem 1 & 2

\[ L_f(X|Y) \]

\[ H_G(X|Y) = \min_{W \in \mathcal{G}} I(W;X|Y) \]

\[ R(0) \]

Conditional Graph Entropy

Lossless

Zero-Distortion

Theorem 1

Theorem 2
Proof of Theorem 1&2

\[
L_f(X|Y) = \min_{W: X \rightarrow W \rightarrow Y} I(W; X|Y)
\]

\[
R(0) = \min_{X \in W \in \Gamma(G)} H_G(X|Y)
\]

Conditional Graph Entropy
Proof of Theorem 1&2
Proof of Theorem 1&2
Proof of Theorem 1&2

Step 1

\[ \geq \]

Lossless

\[ L_f(X|Y) \]

Zero-Distortion

\[ R(0) \]

\[ H_G(X|Y) = \min_{W: X \rightarrow Y} \min_{X \in \mathcal{W} \in \mathcal{G}} I(W; X|Y) \]

Conditional Graph Entropy
Lemma 1: For every $X$, $Y$ and $f$

$$L_f(X|Y) \geq R(0)$$

$$p(Z^N \neq f(X^N,Y^N)) \geq \frac{1}{N} \sum_{i=1}^{N} p(Z_i \neq f(X_i,Y_i))$$

hence every rate achievable with vanishing block error is also achievable with zero distortion
Proof of Theorem 1&2

\[
L_f(X|Y) \geq R(0)
\]

Step 1

\[
H_G(X|Y) = \min_{w \in \Phi(G)} \min_{x \in \mathcal{F}(w)} I(W;X|Y)
\]

Conditional Graph Entropy

Step 2
Step 2

To show

\[
\min_{V \rightarrow X \rightarrow Y} I(V; X|Y) \leq \min_{W \rightarrow X \rightarrow Y} I(W; X|Y)
\]

Need to show that if \( X \in W \in \Gamma(G) \)

Then, there is a function \( g \) over \( \Gamma(G) \times Y \) s.t.

\( f(x, y) = g(w, y) \) whenever \( p(w, x, y) > 0 \)

Hence

\[
E[d(X, Y, g(W, Y))] = 0
\]
Proof of Theorem 1&2

\[ L_f(X|Y) \geq R(0) \geq H_G(X|Y) = \min_{\substack{W \sim X \sim Y \in \mathcal{G}}} I(W;X|Y) \]

Conditional Graph Entropy
Step 3

To show

\[
\min_{\substack{V-X-Y \\ g|Ed(X,Y,g(V,Y)) \leq 0}} I(V;X|Y) \geq \min_{\substack{W-X-Y \\ X \in W \in \Gamma(G)}} I(W;X|Y)
\]

Suppose that V-X-Y and that there exists g such that Ed(X,Y,g(V,Y)) \leq 0. We define W and prove:

1. \( X \in W \in \Gamma(G) \)
2. \( W - X - Y \)
3. \( I(W;X|Y) \leq I(V;X|Y) \)
Outline

- Finish the proof of theorem 1 and 2
- Analysis 2-Way Communication Model
  Definitions
  Results(Theorem 3)
  Proof
Proof of Theorem 1&2

$H_c(X|Y) = \min_{w \rightarrow x \rightarrow y} \min_{x \in W} I(W;X|Y)$

Conditional Graph Entropy

Step 1

$\geq$

Step 2

$\geq$

Step 3

$\leq$

Step 4

Lostless

Zero-Distortion
Step 4

Introduce Robust Typical Set

$$T_{\epsilon}^N(X) = \{ \hat{x} \in X^N | \left| \frac{1}{N} N(x|\hat{x}) - p_x(x) \right| < \frac{\epsilon}{|X|} p_x(x) \}$$

Compare with Strong Typical Set

$$T_{\epsilon}^N(X) = \{ \hat{x} \in X^N | \left| \frac{1}{N} N(x|\hat{x}) - p_x(x) \right| < \frac{\epsilon}{|X|} \}$$
High-level idea
Design a protocol based on conditional graph entropy

Define

\[ J^* = \{ j: (W^j, X) \in T_2 \} \]

\[ K^* = \{ k: (W^k, Y) \in T_3 \text{ and } \Phi(k) = \Phi(J) \} \]

If \( |K^*| = 1 \) then we can decode the message
Step 4

Encoder:
If $J^*$ is empty, transmits an error message. Otherwise, transmits $\Phi(J)$

Decoder:
If $|K^*| \neq 1$, declares an error. Otherwise, proceeds to determine

$$g(W_i^K, Y_i) \text{ for all } i$$
Step 4

Error Analysis:

E1: \[ J^* = \emptyset \]

E2: \[ J^* \neq \emptyset \text{ and } (W^j, Y) \text{ is not in } T_3 \]

E3: \[ J^* \neq \emptyset \text{ and } \exists \, k \neq j \text{ such that } (W^K, Y) \in T_3 \]
   \[ \text{and, } \Phi(k) = \Phi(j) \]

Need to show these errors are exponentially small
Proof of Theorem 1&2
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Outline

- Finish the proof of theorem 1 and 2
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  - Proof
Definitions

Independent instances of a random pair \((X, Y)\)

\[\{(X_i, Y_i)\}_{i=1}^{\infty}\]

A two-message protocol consists of:

- **Y-encoding function** \(\varphi : Y^n \rightarrow \{0,1\}^l\)

- **X-encoding function** \(\varepsilon : \{0,1\}^l \times X^n \rightarrow \{0,1\}^k\)

- **Decoding function** \(\Psi : \{0,1\}^l \times \{0,1\}^k \times X^n \rightarrow Z^n\)
Definitions

Block-error probability

\[ p[\psi(\varphi(Y^n), \varepsilon(\varphi(Y^n), X^n), Y^n) \neq f(X^n, Y^n)] \]

Where

\[ f(X^n, Y^n) = f(X_1, Y_1), f(X_2, Y_2), \ldots, f(X_n, Y_n) \]
Definitions

Rate region

$$R_f^2(X|Y)$$

Two-message communication complexity

$$L_f^2(X|Y) = \min \{ r_x + r_y : (r_x, r_y) \in R_f^2(X|Y) \}$$
Outline

• Finish the proof of theorem 1 and 2

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Theorem 3

Two random variables $U$ and $V$ defined over finite alphabets are admissible if

1) $U \rightarrow Y \rightarrow X$
2) $V \rightarrow UX \rightarrow Y$
3) $U$, $V$, and $Y$ determine $f(X,Y)$
Theorem 3

Theorem 3:

For every \((X,Y)\) and \(f\)

\[
R_f^2(X|Y) = \{r_x + r_y : r_x \geq I(V;X|UY) \}
\]

and \(r_y \geq I(U; Y|X)\) for some admissible \(U\) and \(V\)
Outline

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Proof of Theorem 3

Converse:

1. Obtain an outer bound on \( S_3 = (r_x, r_y, D) \).
Show \( S_3 \subseteq S_3^* \)
Where \( S_3^* \) defined in terms of \( X, Y, T, U, V \)

2. Show \( S_3^* \) can be expressed in terms of only \( X, Y, U, V \)

3. Show \( S_2 \), the 2-D slice with \( D=0 \) is contained in the 2-D region \( S'_2 \)