Quantization Design

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July 28th and 30th, 2014

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Outline

Quantization

Map a Large Set Θ of Input Values to a Smaller set A

- Discrete Quantization : Countable Θ, Countable A
- Continuous Quantization : Uncountable Θ, Countable A
- Scalar Quantization

$$Q : \Theta \rightarrow A \tag{1}$$

- Vector Quantization

$$\mathbf{Q} : \Theta \times \Theta \times \cdots \times \Theta \rightarrow A \tag{2}$$

Problem Setup

N-level Continuous scalar Quantizer

- Source Θ with normalized support $[0, 1]$ and pdf $p(\theta)$
- N-level quantizer $Q_N(\cdot)$
- N Reconstruction levels $\mathcal{A} = \{a_1, \ldots, a_N\}$
- Thresholds $b_1, \ldots, b_{N-1}$ partitioning $[0, 1]$ with $b_0 = 0$ and $b_N = 1$

N-level Continuous scalar Quantizer

- Huffman encode/decode $Q_N(\Theta)$ with rate

$$R = -\sum_{n=1}^{N} P_n \log_2 P_n \tag{3}$$

where

$$P_n \triangleq \int_{b_{n-1}}^{b_n} p(\theta) d\theta \tag{4}$$

- Distortion Metric

$$d : [0,1] \times \mathcal{A} \to R^+ \tag{5}$$

with

$$d(a,a) = 0 \tag{6}$$

- Average Distortion

$$D_N(\mathbf{b}, \mathbf{a}) = \sum_{n=1}^{N} \int_{b_{n-1}}^{b_n} d(\theta, a_n) p(\theta) d\theta \tag{7}$$

Fixed Rate Quantizer Design and Variable Rate Quantizer Design

- N-level Lloyd-Max quantizer : minimize the average distortion for a fixed number of levels N.[1][2]
- N-level Optimum Quantizer : minimize the average distortion for a fixed number of levels N **subject to an entropy constraint of rate**.[3]

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Local Optimality Conditions

- Nearest Neighbor Condition : For fixed reconstruction levels $\{a_k\}$, Given any $\theta \in [a_k, a_{k+1}]$,

$$Q(\theta) = d(\theta, a_k) \leq d(\theta, a_{k+1}) \ ? \ a_k \ : \ a_{k+1} \tag{8}$$

- Centroid Condition : For fixed regions $\{\mathcal{R}_k\}$ with thresholds $\{b_k\}$,

$$a_k = arg \min_a \int_{b_{k-1}}^{b_k} d(\theta, a) p(\theta) d\theta \tag{9}$$

- Zero Probability Boundary Condition, for all b_k, $k = 1, \ldots, K-1$

$$\mathbb{P}(\theta = b_k) = 0 \tag{10}$$

Necessary and Sufficient Conditions

Theorem

The nearest neighbor condition, the centroid condition, and the zero probability of boundary condition are necessary for a Lloyd-Max quantizer to be optimal.

Theorem

If the following conditions hold for a source Θ *and distortion function* $d(\theta, a)$ *:*

1. $p(\theta)$ *is positive and continuous in* $(0, 1)$
2. $\int_0^1 d(\theta, a)p(\theta)d\theta$ *is finite for all* a
3. $d(\theta, a)$ *is zero only for* $\theta = a$*, is continuous in* θ *for all* a*, and is continuous and convex in* a

then the nearest neighbor condition, centroid condition, and zero probability of bound- ary conditions are sufficient to guarantee local optimality of a quantizer.

Outline

Lloyd-Max algorithm

Alternating Minimization

- For fixed $\{a_k\}$, minimize D w.r.t. $\{b_k\}$
- For fixed $\{b_k\}$, minimize D w.r.t. $\{a_k\}$
- D monotone decreasing

Lloyd-Max algorithm

Algorithm 1: Lloyd-Max

Result: Minimize the average distortion for a N-level Lloyd-Max quantizer
step 1) Choose an arbitrary set of initial reconstruction levels $\{a_n\}$
step 2) For each $n = 1, \ldots, N$ set $\mathcal{R}_n = \{\theta | d(\theta, a_n) \leq d(\theta, a_j),\ j \neq n\}$
step 3) For each $n = 1, \ldots, N$ set $a_n = arg \min_a E[d(\Theta, a) | \Theta \in \mathcal{R}_n]$
step 4) Repeat step 2 and 3 until change in average distortion is negligible
step 5) Revise $\{a_n\}$ and $\{\mathcal{R}_n\}$ to satisfy the zero probability of boundary condition

Analysis

- Local optimum guaranteed by Theorem. 2
- Monotonic Convergence in N

$$D_N^*(\mathbf{b}^*, \mathbf{a}^*) = \sum_{n=1}^{N} \int_{b_{n-1}^*}^{b_n^*} d(\theta, a_n^*) p(\theta) d\theta \tag{11}$$

$$D^* = \lim_{N \to \infty} D_N^* \tag{12}$$

Monotonic Convergence in N

The Lloyd-Max N-level quantizer is the solution of the following problem:

$$
\begin{aligned}
D_N^* = \min \sum_{n=1}^{N} \int_{b_{n-1}}^{b_n} & d(\theta, a_n) p(\theta) d\theta \\
s.t.\ & b_0 = 0 \\
& b_N = 1 \\
& b_{n-1} \leq b_n, n = 1, \ldots, N \\
& a_n \leq b_n, n = 1, \ldots, N
\end{aligned} \tag{13}
$$

Monotonic Convergence in N

- Degenerate the N-level Lloyd-Max quantizer to $N-1$
- By adding the additional constraint $b_{N-1} = 1$ to (13) and forcing $a_N = 1$, hence

$$D^*_{N-1} \geq D^*_N \tag{14}$$

- D^*_N bounded below by 0
- D^*_N converges

Outline

Lloyd-Max Quantizer Design Using Dynamic Programming

- For discrete Θ
- One construction level in the interval $(\beta_1, \beta_2) \subseteq [0, 1]$

$$T_1(\beta_1, \beta_2) = \min_a \sum_{\theta \in \Theta \cap (\beta_1, \beta_2)} d(\theta, a) p(\theta) \tag{15}$$

- K construction levels in the interval (β_1, β_2)

$$T_K(\beta_1, \beta_2) = \min_{\mathbf{a}, \mathbf{b}: \beta_1 < b_1 < \cdots < b_{K-1} < \beta_2} \sum_{k=1}^{K} \sum_{\theta \in \Theta \cap (b_{k-1}, b_k)} d(\theta, a) p(\theta) \tag{16}$$

- Notice that

$$D_N^* = T_N(0, 1) \tag{17}$$

Lloyd-Max Quantizer Design Using Dynamic Programming

Theorem

Let $b_1^*, \ldots, b_{K-1}^*$ *be the optimizing boundary points for* $T_K(b_0^* = 0, b_K^* = 1)$*, then* $b_1^*, \ldots, b_{K-2}^*$ *must be the optimizing boundary points for* $T_{K-1}(b_0^*, b_{K-1}^*)$*, and*

$$T_K(b_0^*, b_K^*) = \min_{b_{K-1}: b_0^* < b_{K-1} < b_K^*} [T_{K-1}(b_0^*, b_{K-1}) + T_1(b_{K-1}, b_K^*)] \quad (18)$$

Lloyd-Max Quantizer Design Using Dynamic Programming

- For any $1 < k \leq K$ and any discrete $\beta \in (b_0^*, b_K^*]$

$$T_k(b_0^*, \beta) = \min_{b: b_0^* < b < \beta} [T_{k-1}(b_0^*, b) + T_1(b, \beta)] \tag{19}$$

- Optimizing threshold

$$b_{k-1}^*(b_0^*, \beta) = \underset{b: b_0^* < b < \beta}{arg \min} [T_{k-1}(b_0^*, b) + T_1(b, \beta)] \tag{20}$$

Lloyd-Max Quantizer Design Using Dynamic Programming

Algorithm 2: DP algorithm for Lloyd-Max Quantizer Design

Result: Minimize the average distortion for a N-level Lloyd-Max quantizer

step 1) Compute the values of $T_1(\beta_1, \beta_2)$ for all discrete β_1 and β_2 in $[0, 1]$

step 2) For each $n = 2, \ldots, N$ compute $T_n(0, \beta)$ and $b^*_{n-1}(0, \beta)$ for all β in $(0, 1]$ using (19) and (20)

step 3) Let $b_N = 1$, for each $n = N, \ldots, 2$ set $b_{n-1} = b^*_{n-1}(0, b_n)$

step 4) For each $n = 1, \ldots, N$, set

$$a_k = arg\min_a E[d(\Theta, a)|\Theta \in (b_{k-1}, b_k)]$$

Lloyd-Max Quantizer Design Using Dynamic Programming

Theorem

The boundaries $\{b_k\}_{k=0}^{K}$ and reconstruction levels $\{a_k\}_{k=1}^{K}$ returned by Algorithm 2 represent the optimal quantizer.[4]

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Problem Setup

- Huffman encode/decode $Q_N(\Theta)$ with rate

$$R = -\sum_{n=1}^{N} P_n \log_2 P_n \tag{21}$$

where

$$P_n \triangleq \int_{b_{n-1}}^{b_n} p(\theta) d\theta \tag{22}$$

- Distortion Metric

$$d : [0, 1] \times \mathcal{A} \rightarrow R^+ \tag{23}$$

with

$$d(a, a) = 0 \tag{24}$$

- Average Distortion

$$D_N(\mathbf{b}, \mathbf{a}) = \sum_{n=1}^{N} \int_{b_{n-1}}^{b_n} d(\theta, a_n) p(\theta) d\theta \tag{25}$$

Problem Setup

Minimizing Distortion with a Rate Constraint

$$D_N^* = \min \sum_{n=1}^{N} \int_{b_{n-1}}^{b_n} d(\theta, a_n) p(\theta) d\theta$$
$$s.t. \; R = -\sum_{n=1}^{N} P_n \log_2 P_n \leq H_0 \tag{26}$$

Form the Lagrangian

$$L = D_N(\mathbf{b}, \mathbf{a}) + \lambda(R(\mathbf{b}) - H_0) \tag{27}$$

Outline

Analysis

Take Derivative w.r.t. a_n

$$\begin{aligned}\frac{\partial}{\partial a_n}L &= \frac{\partial}{\partial a_n}D_N(\mathbf{b},\mathbf{a}) \\ &= \int_{b_{n-1}}^{b_n}\left[\frac{\partial}{\partial a_n}d(\theta,a_n)\right]p(\theta)d\theta\end{aligned} \tag{28}$$

Denote the optimum reconstruction levels by

$$a_n^* \triangleq a_n(\mathbf{b}) \tag{29}$$

a_n^* can be obtained by (28), i.e. for the mean-square distortion [2]

$$a_n^* = \frac{\int_{b_{n-1}}^{b_n}\theta p(\theta)d\theta}{\int_{b_{n-1}}^{b_n}p(\theta)d\theta} \tag{30}$$

Analysis

Take Derivative w.r.t. b_n

$$\frac{\partial}{\partial b_n} L = \frac{\partial}{\partial b_n} D_N(\mathbf{b}, \mathbf{a}^*) + \lambda \frac{\partial}{\partial b_n} R(\mathbf{b}) \tag{31}$$

which leads to

$$\lambda(\ln(P_{n+1}/P_n)) = d(b_n, a^*_{n+1}) - d(b_n, a^*_n) \tag{32}$$

for all $n \in \{1, 2, \dots, N-1\}$**.**

Outline

Generalized Max Algorithm

Algorithm 3: Generalized Max Algorithm

Result: Minimize the average distortion for a N-level variable rate optimum quantizer

step 1) Given N and fixed λ, set $b_0 = 0$, choose an initial value for b_1 and set $n = 1$

step 2) For the present values of b_{n-1} and b_n, use (28) and (32) to find b_{n+1}. If $n \leq N - 1$, replace n by $n + 1$ and go to step 2). Otherwise continue

step 3) If b_N obtained in step 2) is equal to 1, the initial guess for b_1 is good and the resulting **b** and **a** satisfy the necessary conditions for optimality. Otherwise go to step 1) and change b_1

Generalized Lloyd Algorithm

Apply the mean square distortion to (32)

$$\lambda(\ln(P_{n+1}/P_n)) = (a^*_{n+1} - a^*_n)(a^*_{n+1} + a^*_n - 2b_n) \tag{33}$$

which can be written as

$$b^*_n = \frac{a^*_{n+1}}{a^*_n} - \frac{\lambda}{2(a^*_{n+1} - a^*_n)} \ln(P_{n+1}/P_n) \tag{34}$$

Generalized Lloyd Algorithm

Algorithm 4: Generalized Lloyd Algorithm

Result: Minimize the average distortion for a N-level variable rate optimum quantizer

step 1) Given N and fixed λ, set $b_0 = 0$, $b_N = 1$, and choose an initial value for $b_1, \dots, b_{N-1}$.

step 2) Compute $\{a_1, \dots, a_N\}$ by (28)

step 3) Compute $\{b_1, \dots, b_{N-1}\}$ by (34)

step 4) Run step 2) and step 3) ℓ times. If D_N^* converges, output **b** and **a**. Otherwise go to step 1) and change initial guess for **b**