Variable Rate Channel Capacity

Jie Ren

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Reference

• This is a introduction of Sergio Verdu and Shlomo Shamai’s paper.

Outline (1st talk)

- Conventional Channel Coding
- Setup of Variable Rate Coding
Conventional Channel Capacity

\[ C = \sup_{p_X} I(X; Y) \]

\[ P(y|x) \]

\[ X \]
Conventional Channel Capacity

• What if Channel state varies?

\[ C = \inf_{m} \sup_{P_X} I(X; Y_m) \]
Example: BSC

- Crossover Probability $p$

$$C = 1 - h(p)$$
Example: BSC

- Crossover Probability $p=0.5$

\[ C = 1 - h(0.5) = 0 \]
Example: BSC

- Crossover Probability $p \sim \text{Uni}(0,0.5)$

\[
C = \inf_{p} 1 - h(p) = 0
\]
Conventional Channel Capacity

• What if Channel state varies?

\[ C = \inf_{m} \sup_{P_{x}} I(X; Y_{m}) \]
Outline

• Conventional Channel Coding
• Setup of Variable Rate Coding
• Variable-to-Fixed Capacity
• Fixed-to-Variable Capacity
• Variable-Blocklength Capacity
• Examples
Setup of Variable Rate Coding

- Fixed-to-fixed coding
- Variable-to-fixed coding
- Fixed-to-variable coding
- Variable-blocklength coding
Fixed-to-fixed

\[ f^{k\to n}: \{0,1\}^k \to x^n \]

\[ g^{n\to k}: y^n \to \{0,1\}^k \]
Fixed-to-fixed

\[
\lim_{n \to \infty} \epsilon_n = 0
\]

\[
\lim_{n \to \infty} \inf_{n}^{k} = R
\]
Variable-to-Fixed

\[ f^n : \{0,1\}^{m_n} \rightarrow x^n \]

\[ g^n : y^n \rightarrow \{0,1\}^{m_n} \]
Variable-to-Fixed

\[ L_n = \max_{(B_1 \ldots B_k) = g^{n:k}(Y^n)} k \]

\[ R = \lim_{n \to \infty} \inf \frac{E[L_n]}{n} \]
Fixed-to-variable

\[ f^k: \{0,1\}^k \to x^\infty \]

\[ \{g^{n:k}: y^n \to \{0,1\}^k\}_{n=1}^\infty \]
Fixed-to-variable

\[ N_k = \min_{g^k(Y^n) = B^k} n \]

\[ [C] = \max_{f, g} \lim_{k \to \infty} \inf \frac{k}{E[N_k]} \]

\[ [\bar{C}] = \max_{f, g} \lim_{k \to \infty} \inf E \left[ \frac{k}{N_k} \right] \]
Variable-blocklength

\[ f^{k^{-}} : \{0,1\}^k \rightarrow x^+ \]

\[ g^{k^{-}} : y^+ \rightarrow \{0,1\}^k \]
Notations

$C$: fixed $\rightarrow$ fixed capacity
$\langle C \rangle$: variable $\rightarrow$ fixed capacity
$[C]$: fixed $\rightarrow$ variable capacity
$[\bar{C}]$: upper fixed $\rightarrow$ fixed capacity
$[[C]]$: variable $\rightarrow$ blocklength capacity
Basic Relationship

\[ C \leq \min ([C], \langle C \rangle) \leq \max ([C], \langle C \rangle) \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_{\epsilon} \]
Example: BSC

\[ P = \begin{cases} 
0.11 & q \\
0 & 1 - q 
\end{cases} \]
Example: BSC

Comparison of capacities for a binary symmetric channel whose crossover probability is 0.11 with probability $q$ and 0 with probability 1.
Variable Rate Channel Capacity

Jie Ren

2013/4/30
Outline (2nd talk)

- Reviews
- Theorems
- Examples
Review: Definitions

- Fixed-to-fixed capacity
  \[
  \lim_{n \to \infty} \epsilon_n = 0
  \]
  \[
  \lim_{n \to \infty} \inf_{n}^{k} = R
  \]
Review: Definitions

• Variable-to-fixed capacity

\[ L_n = \max_{(B_1 \ldots B_k) = g^n:k(Y^n)} k \]

\[ R = \lim \inf_{n \to \infty} \frac{E[L_n]}{n} \]
Review: Definitions

- Fixed-to-variable & upper fixed-to-variable

\[
N_k = \min_{g^n:k(Y^n)=B^k} n
\]

\[
[C] = \max_{f,g} \lim_{k \to \infty} \inf \frac{k}{E[N_k]}
\]

\[
[\bar{C}] = \max_{f,g} \lim_{k \to \infty} \inf E \left[ \frac{k}{N_k} \right]
\]
Review: Definitions

• $\varepsilon$ capacity

\[
\lim_{n \to \infty} \varepsilon_n \leq \varepsilon
\]

\[
\lim_{n \to \infty} \inf_{k} \frac{k}{n} = R
\]
Basic Relationship

\[ C \leq \min ([C], \langle C \rangle) \leq \max ([C], \langle C \rangle) \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_\epsilon \]
State-Dependent Channels

\[ P_{Y^n|X^n}(b^n|a^n) = \sum_{l=1}^{K} \pi_l P_{Y^n|X^n,S}(b^n|a^n, l) \]
State-Dependent Channels

- Finite alphabets

\[ P_{Y^n|X^n,S}(b^n|a^n,l) = \prod_{i=1}^{n} W_l(b_i|a_i) \]
Outline (2\textsuperscript{nd} talk)

- Reviews
- Theorems
- Examples
Theorem 7

- Two possible states

\[ P_{Y^n|X^n}(b^n|a^n) = \pi_1 \prod_{i=1}^{n} W_1(b_i|a_i) + \pi_2 \prod_{i=1}^{n} W_2(b_i|a_i) \]

- Variable-to-fixed capacity in SDC

\[ \langle C \rangle = \max_{P_{XU}} \{ \min \{ I(U, W_1), I(U, W_2) \} \}
+ \max \{ \pi_1 I(X, W_1 | U), \pi_2 I(X, W_2 | U) \} \]
Theorem 12

• Fixed-to-fixed capacity in SDC

\[ C = \max_P \min_l I(P, W_l) \]
Theorem 13

- Fixed-to-variable capacity in SDC

\[
[C] \geq \left( \min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1}
\]
Theorem 14

• Suppose

$$C_l = \max_{P_X} I(P_X, W_l) = I(P^*_X, W_l)$$

• Then the fixed-to-variable capacity in SDC

$$[C] = \left( \sum_{l=1}^{K} \frac{\pi_l}{C_l} \right)^{-1}$$
Theorem 15

- Upper-fixed-to-variable capacity in SDC

\[
[\bar{C}] = \max_{P_X} \sum_{l=1}^{K} \pi_l I(P_X, W_l)
\]
Outline (2nd talk)

- Reviews
- Theorems
- Examples
Example 1

• With probability $q$

• With probability $1-q$

$C_0$
Example 1

• By definition

\[ C = [C] = 0 \]

\[ \lim_{\epsilon \uparrow 1} C_{\epsilon} = C_0 \]

\[ \langle C \rangle = [\bar{C}] = qC_0 \]
Example 2

- With probability $\pi_0 : \text{BSC}(\delta_0)$
- With probability $\pi_1 : \text{BSC}(\delta_1), 0.5 > \delta_1 > \delta_0$
Example 2

\[ P = \begin{pmatrix} \delta_0 & \pi_0 \\ \delta_1 & \pi_1 \end{pmatrix} \]
Example 2

• By theorem 12

\[ C = 1 - h(\delta_1) \]
Example 2

- By theorem 7

\[
\langle C \rangle = 1 - \pi_0 h(\delta_0) - \min_{0 \leq \gamma \leq 1} h(\gamma \ast \delta_1) - \pi_0 h(\gamma \ast \delta_0)
\]

\[
\gamma \ast \delta = (1 - \delta)\gamma + (1 - \gamma)\delta
\]
Example 2

• By theorem 14 & 15

\[
[C] = \left(\frac{\pi_0}{1 - h(\delta_0)} + \frac{\pi_1}{1 - h(\delta_1)}\right)^{-1}
\]

\[
[C] = 1 - \pi_0 h(\delta_0) - \pi_1 h(\delta_1)
\]
Example 3

• Channel alphabet \{0...1023\}
• With probability 0.9: no error
• With probability 0.1: \( y_i=(x_i>0) \)
Example 3

• By theorem 13

\[ [C] \geq \left( \min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1} = 4.4 \]
Example 3

- Suboptimal scheme

\[
N_k = \begin{cases} 
\frac{k}{10} & 0.9 \\
\frac{11k}{10} & 0.1 
\end{cases}
\]

\[
\frac{k}{E[N_k]} = 5
\]
Example 4

• Binary erasure channel \( \text{BEC}(E) \)
  – \( E \) random variable
  – Stay constant during the duration of the codeword

\[
F(x) = P[1 - E \leq x]
\]
Example 4

\[ [C] = (E[\frac{1}{1 - E}])^{-1} = (\int_{1}^{\infty} F(\frac{1}{x})d\alpha)^{-1} \]

\[ [\bar{C}] = 1 - \int_{0}^{1} F(x)dx \]

\[ \langle C \rangle = \max_{0 \leq s \leq 1} s - sF(s) \]
Example 4

- $E \sim \text{Uni}(0,1)$

\[
C = [C] = 0 \\
\langle C \rangle = \frac{1}{4} \\
[C] = \frac{1}{2}
\]
Example 5

• Gaussian channel with nonergodic fading

\[ y_i = H x_i + n_i \]
\[ H \in [H_{\text{min}}, H_{\text{max}}] \]
\[ \inf\{x: F_H(x) > 0\} = H_{\text{min}} \]
\[ \inf\{x: F_H(x) = 1\} = H_{\text{max}} \]
\[ \frac{1}{n} \sum_{i=1}^{n} |x_i|^2 \leq P \]
Example 5

• Gaussian channel with nonergodic fading

\[ C = \log(1 + H_{min}P) \]

\[ \lim_{\epsilon \uparrow 1} C_\epsilon = \log (1 + H_{max}P) \]
Example 5

- Gaussian channel with nonergodic fading

\[
[C] = \left( E \left[ \frac{1}{\log(1 + HP)} \right] \right)^{-1}
\]

\[
[C] = E \left[ \log \left( 1 + HP \right) \right]
\]
Example 5

- Gaussian channel with nonergodic fading

\[
\langle C \rangle = \int_{x_0}^{x_1} (1 - F_H(x)) \left( \frac{f_H(x)}{f_H(x)} + \frac{2}{x} \right) dx \log e
\]
Variable Rate Channel Capacity

Jie Ren

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Outline (3\textsuperscript{rd} talk)

• Theorems and proofs
Theorems and Proofs

\[ C \leq \min ([C], \langle C \rangle) \leq \max ([C], \langle C \rangle) \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_\epsilon \]
Theorem 1

• For all channels

\[ C \leq \langle C \rangle \]
Proof

• Consider a fixed-to-fixed code

\[ \lim_{n \to \infty} \epsilon_n = 0 \]

\[ \lim_{n \to \infty} \inf_{n}^{k} = R \]

• Let

\[ f^{n} = f^{k \to n}(b_{1:k}) \]

\[ g^{n}(y^{n}) = (g^{n \to k}(y^{n}), 0, \ldots, 0) \]
Theorem 2

• If the channel input alphabet is finite

\[ \langle C \rangle \leq \lim_{\epsilon \uparrow 1} C_\epsilon \]
Proof

• Prove by contradiction

\[ C_2 = \frac{1}{2} \langle C \rangle + \frac{1}{2} \lim_{\epsilon \uparrow 1} C_\epsilon \]

\[ P \left[ \frac{L_n}{n} \leq C_2 \right] \leq \epsilon_0 \]
Theorem 3

• State-dependent channel, suppose each component satisfies strong converse

\[ \langle C \rangle \leq \sum_{i=1}^{K} \pi_i C_i \]
Proof

• The bound would be tight if the encoder knows the channel state
• Hence it’s an upper bound in general
Theorem 4

• Capacity region or 2-user DMBC with degraded message sets

\[ C = \bigcup_{P_{X,U}} \left\{ (R_1, R_2) : \begin{array}{l}
R_1 \leq \min\{I(U, Y_1), I(U, Y_2)\} \\
R_2 \leq I(X; Y_2 | U) \end{array} \right\} \]
Proof

- Thomas M. Cover, “Elements of Information Theory”, page 568
Theorem 5

• In state-dependent channels

\[ \langle C \rangle \geq \max_P \max_{(R_1:K) \in \mathcal{C}^K} \sum_{j=1}^{K} R_j \sum_{i=j}^{K} \pi_{P^{-1}(i)} \]
Proof

• Assume identity permutation
• Show

\[ \langle C \rangle \geq \sum_{j=1}^{K} R_j \sum_{i=j}^{K} \pi_{(i)} \]
Theorem 6

• Theorem 5 holds with equality if the K-user broadcast channels satisfy the strong converse.
Proof

• Number the channel states in order of increasing expected number of recovered bits

\[ r_i = \lim_{n \to \infty} \inf \frac{k_i}{n} \]
Proof

• For an arbitrarily small $\delta$

\[
R_1 = r_1 - \delta \\
R_i = r_i - r_{i-1}
\]
Proof

• Hence,

\[ R_{1:K} \text{ is } \epsilon - \text{achievable} \]

\[ \xrightarrow{\text{Strong Converse}} \quad R_{1:K} \text{ is achievable} \]
Theorem 7

• Suppose the channel has finite input/output alphabets and is memoryless.

\[
\langle C \rangle = \max_{P_{XU}} \{\min\{I(U, W_1), I(U, W_2)\} \\
+ \max\{\pi_1 I(X, W_1 | U), \pi_1 I(X, W_2 | U)\}\}
\]
Proof

• When the constituent channels are discrete memoryless, the broadcast channel with degraded message sets satisfies the strong converse.

• Theorem 7 follows from 4 and 6
Theorem 8

• For all channels

\[ C \leq [\bar{C}] \]
Proof

• Consider a fixed-to-fixed code

\[ \lim_{n \to \infty} \epsilon_n = 0 \]

\[ \lim_{n \to \infty} \inf_{n \leq k} n = R \]

• Define fixed-to-variable code

\[ f^k: = (f^{k \to n}(b_{1:k}), a, a, ...) \]
Theorem 9

• Suppose the channel has finite memory

\[ C \leq [C] \]
Proof

• Similar proof as theorem 8
Theorem 10

• Assume either the input or output alphabets are finite

\[ [C] \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_\epsilon \]
Proof

• Jensen’s inequality
• Counterpart of Theorem 2
Theorem 11

• For all channels

$$\langle C \rangle \leq [\bar{C}]$$
Proof

• Assume

\[
\lim_{k \to \infty} \inf E \left[ \frac{k}{N_k} \right] = \lim_{k \to \infty} \sup E \left[ \frac{k}{N_k} \right]
\]
Proof

- Non-anticipatory setting
- Compatible variable-to-fixed encoder

\[ f^{(n+1)}(b_1, \ldots, b_{m_n}) = (f^{n}(b_1, \ldots, b_{m_n}), A) \]
Proof

• Construct a sequence of rateless encoders
  – Let the ith component be equal to the ith component of variable-to-fixed code
  – For n sufficiently large
Proof

\[ N_{L_i} \neq i \]

\[ N_{L_i} = \lfloor f^{-1}(\lfloor f(i) \rfloor) \rfloor \]
Proof

\[ N_{L_i} \leq i \]

\[ \frac{L_n}{n} \leq \frac{L_n}{N_{L_n}} \]
Theorem 12

• In state-dependent channel

\[ C = \max_P \min_l I(P, W_l) \]
Proof

• Shannon capacity
• Has to satisfy the worst channel
Theorem 13

• In state-dependent channel

\[
[C] \geq \left( \min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1}
\]
Proof

• Achievability proof

• Scheme:
  – Fixed-to-variable
  – Repition after the first transmission
Theorem 14

• In state-dependent channel, suppose

\[ C_l = \max_{P_X} I(P_X, W_l) = I(P_X^*, W_l) \]

• Then theorem 13 holds with equality

\[ [C] = \left( \min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1} \]
Proof

- Converse proof
- Data-processing inequality

\[
\frac{E[N_{K}]}{K} \geq \sum_{s=1}^{K} \frac{\pi_s}{C_s}
\]

\[
\frac{K}{E[N_{K}]} \leq \left( \sum_{s=1}^{K} \frac{\pi_s}{C_s} \right)^{-1}
\]
Theorem 15

• In state-dependent channel

\[
[C] = \max_{P_X} \sum_{l=1}^{K} \pi_l I(P_X, W_l)
\]
Proof

• Achievability
  – Assume the decoder has knowledge of the ergodic mode
  – Apply theorem 8

\[
[\tilde{C}] \geq C = \max_{P_X} \sum_{l=1}^{K} \pi_l I(P_X, W_l)
\]
Proof

- Converse

\[ \frac{1}{n} I(B_{1:k}; \hat{B}_{1:k}) \leq \frac{1}{n} I(f^{k:}(B_{1:k}); Y^n) \leq I(Q_x, W_l) \]

\[ E \left[ \frac{k}{N_k} \mid S = l \right] \leq I(Q_x, W_l) \]