

A new computational approach for determining rate regions and optimal codes for coded networks

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Abstract—A new computational technique is presented for determining rate regions for coded networks. The technique directly manipulates the extreme ray representation of inner and outer bounds for the region of entropic vectors. We use new inner bounds on region of entropic vectors based on conic hull of ranks of representable matroids. In particular, the extreme-ray representations of these inner bounds are obtained via matroid enumeration and minor exclusion. This is followed by a novel use of iterations of the double description method to obtain the desired rate regions. Applications in multilevel diversity coding systems (MDCS) are discussed as an example. The special structure of the problem that makes this technique inherently fast along with being scalable is also discussed. Our results demonstrate that for each of the 31 2-level 3-encoder and the 69 3-level 3-encoder MDCS configurations, if scalar linear codes (over any field) suffice to achieve the rate region, then in fact binary scalar linear codes suffice. For the 31 2-level 3-encoder cases where scalar codes are insufficient we demonstrate that vector linear codes suffice and provide some explicit constructions of these codes.

I. INTRODUCTION

The rate region of multilevel diversity coding systems (MDCS) [13], [6], [14] and coded networks [15] can be expressed in terms of the closure of the region of entropic vectors [15]. Since the closure of the region of entropic vectors $\bar{\Gamma}_N^*$ is unknown for $N \geq 4$, this expression is evaluated by substituting inner and outer bounds for $\bar{\Gamma}_N^*$, thus yielding inner and outer bounds on the rate regions. When these inner and outer bounds for the rate region coincide, the rate region of that network has been determined. This paper makes use of inner bounds based on conic hull of rank functions of representable matroids, because representations of matroids provide optimal codes to achieve the whole rate region as well.

In previous work on connections between matroids and network codes [1], all source entropies are the same and all edge capacities are the same, and hence the capacity of a network is thus a scalar. In contrast, in this paper, the object under study is rate regions, and hence source entropies and edge capacities are allowed to be unequal. Further, while we generalize the mapping in [1] between networks and matroids, to allow one random variable to be mapped to a subset, rather than singleton, of ground set of a matroid. This allows more versatile codes based on subspaces to be used and hence provides a larger achievable rate region.

The primary contribution of this paper is a new computational technique to obtain the rate region of MDCS. The method utilizes non-isomorphic matroid enumeration [10],

together with forbidden minor characterizations for binary and ternary representability [12], to obtain extreme ray representations of the cones generated by *i*) matroids, *ii*) binary representable matroids, and *iii*) dimensions of subspace arrangements in a manner that exploits symmetries. These inner bounds for the region of entropic vectors are then intersected with equalities and inequalities associated with the MDCS problem under study. The extreme ray representation of the intersection is obtained via a novel application of the double description method [4] used in polyhedral representation conversion.

The technique used in our previous work [9] to solve the same problem works with the inequality representations of bounds, and does not exploit the high degree of inherent symmetry in the problem, and thus suffers from an unnecessarily higher computational complexity. In contrast, the technique discussed in this paper works with the extreme ray representation of inner and outer bounds of $\bar{\Gamma}_N^*$, allowing symmetry to be exploited and the complexity to be reduced.

The new method allows us to improve upon the results in our prior work [9]. We consider each of the 31 2-level 3-encoder and 69 3-level 3-encoder MDCS configurations. We show that 25 (54) of the 31 (69) 2-level (3-level) rate regions are achievable by scalar linear codes, in fact binary linear codes suffice. We show that for each of the 6 of 31 2-level cases where scalar linear codes are insufficient that vector linear codes are sufficient, and we provide explicit constructions for some examples.

Further, Hau [6] showed superposition coding was insufficient to achieve the rate region for 12 of the 69 cases, but did not provide an efficient coding scheme in 8 of these 12 cases. In fact, 5 of these 8 cases are in the list of 54 cases for which we have shown binary codes suffice. That is, our construction has produced binary coding constructions that are strictly better than superposition coding for these cases.

The paper is organized as follows. §II introduces bounds on region of entropic vectors. §III defines the rate regions, and §IV deals with the new procedure to compute them and why it is efficient. In §V we show how to construct codes from the bounds, while in §VI we review results obtained for specific MDCS problems. §VII concludes the paper.

II. NEW BOUNDS ON $\bar{\Gamma}_N^*$

In this section, we discuss region of entropic vectors, Shannon outer bound, new representable matroid inner bounds

and dimension functions of subspace arrangements.

A. Shannon outer bound Γ_N on entropic vectors region $\bar{\Gamma}_N^*$

Let (X_1, \dots, X_N) be a collection of N discrete random variables with joint probability mass function $p_{\mathbf{X}}$. To each of the $2^N - 1$ non-empty subsets of the collection, $X_{\mathcal{A}} := (X_i | i \in \mathcal{A})$ with $\mathcal{A} \subseteq \{1, \dots, N\}$, there is an associated joint Shannon entropy $H(X_{\mathcal{A}})$. Stacking these subset entropies for different subsets into a $2^N - 1$ dimensional vector we form a vector \mathbf{h} that lies in $\mathbb{R}^{2^N - 1}$, and is said to be *entropic*. Letting \mathcal{P}_N denote the set of all such possible joint probability mass functions $p_{\mathbf{X}}$, we define $\Gamma_N^* := \mathbf{h}(\mathcal{P}_N)$, the region of entropic vectors, to be the image of \mathcal{P}_N under this function $\mathbf{h}(\cdot)$. It is known that the closure of the region of entropic vectors $\bar{\Gamma}_N^*$ is a convex cone [15]. Viewed as a function of the subset of variables selected, Shannon entropy is a non-decreasing submodular set function, yielding the Shannon outer bound

$$\Gamma_N := \left\{ \mathbf{h} \in \mathbb{R}^{2^N - 1} \mid \begin{array}{l} h_A \leq h_B \quad \forall \mathcal{A} \subseteq \mathcal{B} \\ h_{C \cup D} + h_{C \cap D} \leq h_C + h_D \quad \forall \mathcal{C}, \mathcal{D} \end{array} \right\}$$

While $\Gamma_2 = \Gamma_2^*$ and $\Gamma_3 = \bar{\Gamma}_3^*$, $\bar{\Gamma}_N^* \subsetneq \Gamma_N$ for all $N \geq 4$ [15], it is known [3] that $\bar{\Gamma}_N^*$ is not even polyhedral for $N \geq 4$.

B. Representable matroid inner bounds for $\bar{\Gamma}_N^*$

Definition 1. A *matroid* [12] on a ground set \mathcal{S} of size $|\mathcal{S}| = N$ can be defined via its *rank function*, which is a function $r: 2^{\mathcal{S}} \rightarrow \{0, \dots, N\}$ obeying for all $A, B \subseteq \mathcal{S}$:

- 1) Cardinality: $r(A) \leq |A|$;
- 2) Submodularity: $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.
- 3) Monotonicity: if $A \subseteq B \subseteq \mathcal{S}$ then $r(A) \leq r(B)$.

Observe that 2) and 3) are the same set function requirements as those defining the Shannon outer bound Γ_N . We denote the conic hull of all ranks of matroids with ground set of N elements as Γ_N^{mat} . Partially owing to the large amount of symmetry, and partially to simple combinatorial explosion, the number of matroids grows rapidly with dimension. One can remove a part of the explosion due to the symmetries by working with lists of *non-isomorphic* matroid rank functions, which have long been available for $N \leq 8$ and have recently become available for $N = 9$ [10].

Representable matroids are the class of matroids whose rank functions are in fact ranks of subsets of columns of a matrix. In particular, a matroid M with ground set \mathcal{S} of size $|\mathcal{S}| = N$ and rank $r(\mathcal{S}) = k$ is representable over the finite field \mathbb{F}_q of size q if there exists a matrix $\mathbf{A} \in \mathbb{F}_q^{k \times N}$ such that $\forall B \subseteq \mathcal{S}$ $r(B) = \text{rank}(\mathbf{A}_{:,B})$, the matrix rank of the columns of \mathbf{A} indexed by B . Let Γ_N^q be the conic hull of all rank functions of matroid with N elements and representable in \mathbb{F}_q . This provides an inner bound $\Gamma_N^q \subseteq \bar{\Gamma}_N^*$, because any extremal rank function r of Γ_N^q is by definition representable and hence is associated with a matrix representation $\mathbf{A} \in \mathbb{F}_q^{k \times N}$, from which we can create the random variables

$$(X_1, \dots, X_N) = \mathbf{u}\mathbf{A}, \quad \mathbf{u} \sim \mathcal{U}(\mathbb{F}_q^k). \quad (1)$$

whose elements are $h_A = r(A) \log_2 q$, $\forall A \subseteq \mathcal{S}$. Hence, all extreme rays of Γ_N^q are entropic, and $\Gamma_N^q \subseteq \bar{\Gamma}_N^*$.

One can further generalize the relationship between representable matroids and entropic vectors established by (1) by partitioning $\mathcal{S} = \{1, \dots, N\}$ up into N' disjoint sets, $\mathcal{S}_1, \dots, \mathcal{S}_{N'}$ and defining for $n' \in \{1, \dots, N'\}$ the new vector valued, random variables $\mathbf{X}'_{n'} = [X_n | n \in \mathcal{S}_{n'}]$. The associated entropic vector will have entropies $h_A = r(\cup_{n' \in A} \mathcal{S}_{n'}) \log_2 q$, and is thus proportional to a *projection* of the original rank vector \mathbf{r} keeping only those elements corresponding to all elements in a set in the partition appearing together. Thus, such a projection of Γ_N^q forms an inner bound to $\bar{\Gamma}_{N'}^*$, which we will refer to as a *vector representable matroid inner bound*. The union over all such projections and field sizes for vector representable matroid inner bounds is the conic hull of the set of ranks of subspaces, which will be discussed next.

C. Dimension function of subspace arrangements

Consider a collection of N vector subspaces $\mathcal{V} = (V_1, \dots, V_N)$ of a finite dimensional vector space, and define the set function $d: 2^{\mathcal{V}} \rightarrow \mathbb{N}_+$, where $d(A) = \dim(\sum_{i \in A} V_i)$ for each $A \subseteq [N]$ is the dimension of the vector space generated by the union of subspaces indexed by A . For any collection of subspaces \mathcal{V} , the function d is integer valued, and obeys monotonicity and submodularity. Additionally, for every subspace dimension function d , there is an associated entropic vector. Indeed, one can place the vectors forming a basis for each V_i , over all i , side by side into a matrix \mathbf{A} , which when utilized in (1), will yield random subvectors having the desired entropies, as discussed at the end of §II-B. Thus, the conic hull of dimensions of subspace arrangements forms an inner bound on $\bar{\Gamma}_N^*$, we denote it by Γ_N^{space} .

Integrality, monotonicity, and submodularity are necessary but insufficient for for a given set function $d: 2^{\mathcal{V}} \rightarrow \mathbb{N}_+$ to be dimension function of subspace arrangements. That is, there exist additional inequalities that are necessary to describe the conic hull of all possible subspace dimension set functions. As discussed in [5], Ingleton's inequality [7] together with the Shannon outer bound Γ_4 , completely characterizes Γ_4^{space} .

For $N = 5$ subspaces [2] found 24 new inequalities in addition to the Ingleton inequalities that hold, and prove this set is irreducible and complete in that all inequalities are necessary and no additional non-redundant inequalities exist. For $N \geq 6$, [2], [8] there are new inequalities from $N - 1$ to N , and Γ_N^{space} remains unknown.

III. MDCS CODES AND THEIR RATE REGIONS

Here, we review the definition and structure for rate regions for MDCS style networks. There are K *independent* sources $X_{1:K} \equiv (X_1, \dots, X_K)$ where source k has normalized entropy in \mathbb{F}_q , that is $H_q(X_k) = \sum_{x \in \mathcal{X}_k} -p_x \log_q(p_x)$ where \mathcal{X}_k is the support for X_k . Each source X_k is in fact a sequence of random variables $\{X_k^t, t = 1, 2, \dots\}$ i.i.d. in t , so that X_k is a representative random variable with this distribution.

All sources are made available to each of a collection of encoders indexed by the set E , the outputs of which are description/message variables $U_e, e \in E$. The message variables are mapped to a collection of decoders index by the set D ,

where a decoder must losslessly (in the typical Shannon sense) recover a subset of source variables $X_{\beta(d)}$, $d \in D$, $\beta(d) \subseteq [K]$.

Let $\mathbf{R} = (R_1, \dots, R_{|E|}) \in \mathbb{R}_+^E$ be the rate vector for the encoders, where each R_e is also calculated in \mathbb{F}_q . An (n, \mathbf{R}) block code in \mathbb{F}_q for each encoder e is defined by

$$f_e : \prod_{i=1}^n \prod_{k=1}^K \mathcal{X}_k \rightarrow \{0, 1, \dots, [q^{nR_e}]\}, \quad e \in E. \quad (2)$$

The blocked encoder outputs are indicated by $U_e = f_e(X_{1:K}^{1:n})$ for $e \in E$. Decoder d has an available subset of the descriptions $\mathcal{E}_j \subseteq E$, and must asymptotically losslessly recover source variables $X_{\beta(d)}$:

$$g_d : \prod_{e \in \mathcal{E}_d} \{1, \dots, [q^{nR_e}]\} \rightarrow \prod_{i=1}^n \prod_{k \in \beta(d)} \mathcal{X}_k, \quad d \in D. \quad (3)$$

A rate vector \mathbf{R} is *achievable* if there exists, for a sufficiently large n , an (n, \mathbf{R}) block code in \mathbb{F}_q such that the coding rate for each encoder $\hat{R}_e \leq R_e$, $e \in E$ and $\mathbb{P}(g_d(f_e(X_{1:K}^{1:n})), e \in \mathcal{E}_d) \neq X_{\beta(d)} \rightarrow 0$ as $n \rightarrow \infty$. The rate region \mathcal{R} is the collection of all achievable rate vectors.

An outer bound on coding rate region \mathcal{R} for MDSCS is

$$\mathcal{R}_{\text{out}} = \text{Ex}(\text{proj}_{h_{U_e, e \in [E]}}(\bar{\Gamma}_N^* \cap \mathcal{L}_{0123})) \quad (4)$$

and a similar expression for an inner bound on the rate region also exists [16], [15]. Here, $\mathcal{L}_{0123} = \cap_{i=0}^3 \mathcal{L}_i$ and \mathcal{L}_i denote various linear constraints that are derived from the network, $\text{proj}_{h_{U_e, e \in [E]}}(\mathcal{B})$ is the projection of set \mathcal{B} on coordinates $(h_{U_e} | e \in [E])$ corresponding to encoder output variables, and $\text{Ex}(\mathcal{B}) = \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : \mathbf{h} \geq \mathbf{h}' \text{ for some } \mathbf{h}' \in \mathcal{B}\}$.

For $N \geq 4$, the exact region of entropic vectors $\bar{\Gamma}_N^*$ is unknown, hence, one must replace $\bar{\Gamma}_N^*$ in (4) with an outer bound or inner bound to obtain a corresponding outer or inner bound on the rate region. When these bounds match, they are both equal to the fundamental rate region. Otherwise, the exact rate region lies between inner and outer bound.

IV. NEW COMPUTATIONAL APPROACH

The bounds for $\bar{\Gamma}_N^*$ discussed in §II are polyhedral. Hence, if we replace $\bar{\Gamma}_N^*$ in (4) by these bounds, rate region calculations reduce to the generic problem of applying linear constraints to a polyhedron and then projecting it down to a subset of dimensions. The same reduction can be used to obtain rate regions for network coding in directed acyclic networks [15], as well as other coding problems [16].

Our computational approach is inspired by the double description (DD) method of polyhedral representation conversion. Given the inequality representation of polyhedral cone, the DD method incrementally constructs its extreme ray representation by applying each inequality constraints one at a time. As we shall see in this and next section we can apply constraints by using the framework for an iteration of the double description method.

A. An Iteration of Double Description Method

The DD Method, first described in [11], enumerates all extreme rays of a pointed polyhedral cone with the origin as its only extreme point. As all polyhedra can be converted to cones

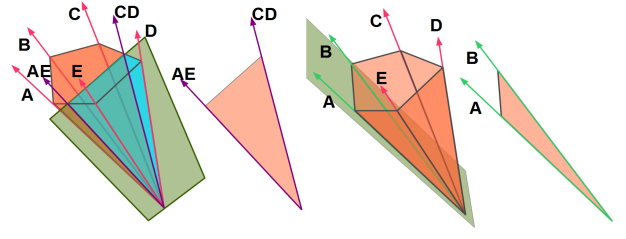


Figure 1. Depiction of special structure of the problem: (top row) Insertion of a generic equality constraint (green) using corollary 5 where rays A, E and C, D are combined to compute new rays AE and CD and (bottom row) Insertion of \mathcal{L}_i in any polyhedral cone $\mathcal{P} \subset \Gamma_N$ obviating any computation.

through homogenization, it follows that conic conversion constitutes the general case [17]. Fukuda *et al.* discuss efficient implementations of DD in [4]. We give some definitions.

Definition 2. A polyhedral cone \mathcal{P} has dual representations as *i*) an intersection of halfspaces intersecting the origin $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \geq \mathbf{0}\}$ and *ii*) the conic hull of its extreme rays $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \mathbf{G}\boldsymbol{\lambda} : \boldsymbol{\lambda} \geq \mathbf{0}\}$. \mathbf{A} is called the representation matrix, while \mathbf{G} is called the generator matrix.

Definition 3. A pair (\mathbf{A}, \mathbf{G}) is said to be a *double description pair* (DD pair) if the relationship: $\mathbf{A}\mathbf{x} \geq \mathbf{0}$ iff $\mathbf{x} = \mathbf{G}\boldsymbol{\lambda}$ for some $\boldsymbol{\lambda} \geq \mathbf{0}$ holds. Here \mathbf{A} has d columns and \mathbf{G} has d rows.

Lemma 4 below is at the core of DD. Given a DD pair (\mathbf{A}, \mathbf{G}) corresponding to a polyhedral cone $\mathcal{P} \subset \mathbb{R}^d$, it describes how to obtain generator matrix \mathbf{G}' corresponding to polyhedral cone $\mathcal{P}' \subset \mathbb{R}^d$ defined by representation matrix $\begin{pmatrix} \mathbf{A} \\ \mathbf{a}^T \end{pmatrix}$. We denote by H^+ , H^0 and H^- the partition of \mathbb{R}^d produced by insertion of a new inequality $\mathbf{a}^T \mathbf{x} \geq 0$ as defined in [4]. The rays in \mathbf{G} can then be partitioned as J^+ , J_0 and J^- , respectively, based on their membership in one of the three partitions above.

Lemma 4. The Double Description Lemma: Let (\mathbf{A}, \mathbf{G}) be a DD pair corresponding to cone $\mathcal{P} \subset \mathbb{R}^d$ and $\mathbf{a}^T \mathbf{x} \geq 0$ be a halfspace in \mathbb{R}^d . Then the pair $(\mathbf{A}', \mathbf{G}')$ is a DD pair, where \mathbf{G}' is the $d \times |J'|$ matrix with column vectors \mathbf{r}_j ($j \in J'$) defined by $J' = J^+ \cup J_0 \cup (J^+ \times J^-)$, and $\mathbf{r}_{jj'} = (\mathbf{a}^T \mathbf{r}_j) \mathbf{r}_{j'} - (\mathbf{a}^T \mathbf{r}_{j'}) \mathbf{r}_j$ for each $(j, j') \in J^+ \times J^-$.

We extend Lemma 4 for equality constraints as follows:

Corollary 5. If \mathbf{G} is a $d \times n$ generator matrix for cone \mathcal{P} and $\mathbf{a}^T \mathbf{x} = b$ is a hyperplane then $\mathcal{P} \cap \{\mathbf{a}^T \mathbf{x} = b\}$ is a cone whose generator matrix \mathbf{G}' is $d \times |J'|$ matrix with column vectors \mathbf{r}_j ($j \in J'$) defined by $J' = J_0 \cup (J^+ \times J^-)$ and $\mathbf{r}_{jj'} = (\mathbf{a}^T \mathbf{r}_j) \mathbf{r}_{j'} - (\mathbf{a}^T \mathbf{r}_{j'}) \mathbf{r}_j$ for each $(j, j') \in J^+ \times J^-$.

It is clear from Lemma 4 (Corollary 5) that one does not require knowledge of \mathbf{A} to compute \mathbf{G}' from \mathbf{G} when inserting a new inequality (equality) constraint.

B. A Note on Complexity and structure of the problem

In terms of time complexity, in general, the computation of $\mathbf{r}_{jj'}$ while applying each constraint can be onerous. Starting with a $d \times m$ matrix \mathbf{G} , we need $O(m)$ time for testing each

ray's membership in J^0 , J^+ or J^- . Then we need $O(m^2)$ time to compute all $r_{jj'}$. Hence the overall time complexity for applying constraints is $O(m^2)$ time. This bound holds for equality as well as inequality constraints.

However, it should be noted that most of the equality constraints introduced when calculating network rate regions, are already sign definite.

Theorem 6. *Let a polyhedral cone $\mathcal{P} \subseteq \Gamma_N$, $\forall L \in \mathcal{L}_i, i = 0, 1, 2, 3$, $J^- = \emptyset$ in $\mathcal{P} \cap L$.*

Proof: Note that these constraints are conditional mutual information (entropies) equal to zero. If $J^- \neq \emptyset$, there exists some point $p \in \mathcal{P}$ such that $L(p) < 0$. However, $p \in \Gamma_N$, so it satisfies Shannon inequalities which are non-negativity of conditional mutual information (entropy). Contradiction. ■

Thus, the growth in computation and the introduction of new extreme rays is avoided due to this sign definite structure, as shown in Fig. 1. The time complexity of applying one constraint in this case is hence $O(m)$, allowing the proposed method to execute quickly. Furthermore, this advantage is independent of the size of the problem under consideration, making the proposed technique inherently scalable.

The computational approach in [9] uses an inequality for the region bounds. Though applying constraints to an inequality representation of a polyhedron takes constant time, the subsequent Fourier Motzkin variable elimination utilized in [9] suffers from doubly exponential time complexity. In our experiments this complexity resulted in excessively long running times for experiments with $N \geq 6$ variables. This motivated us to find a more efficient way to compute the rate region, especially for large N . One of the merits of working with an extreme ray representation is that projection is easy: one may merely delete the coordinates to be removed under projection, then remove any resulting redundant projected rays.

C. Proposed computation approach

The proposed approach to obtain the scalar (binary) representable matroid inner bounds for rate regions are as follows:

- 1) Start with a list of rank functions of non-isomorphic matroids, exclude those containing a forbidden minor (e.g., for binary representable remove those having a $U_{2,4}$ minor). Add all permutations of the remaining non-isomorphic matroid ranks to get a full list of all ranks for representable matroids on a ground set size of N .
- 2) Use one iteration of DD method to apply topology constraints and rate constraints from the network graph.
- 3) Projection: drop all coordinates except those corresponding to source and coding rate variables from each ray.
- 4) Remove any redundant rays (i.e., conic hull) to get an extreme ray representation for the rate region. Additionally, representation conversion is necessary if an inequality representation of the rate region is desired.

V. CONSTRUCTING CODES FROM Γ_N^q AND Γ_N^{SPACE}

Regions built from representable matroids not only provide inner bounds on $\bar{\Gamma}_N^*$, but also allow the determination of

linear network codes via their matrix representations. In this section, we explain how any point in the rate region obtained from the representable matroid inner bound can be achieved with a linear code constructed from matrix representations of representable matroids.

Let \mathcal{R}_q be the rate region obtained from Γ_N^q . Note that \mathcal{R}_q is a cone in dimension of $[h_{X_k}, h_{U_e}, k \in [K], e \in E]$. Let $\text{Extr}(\mathcal{P})$ be the set of representative vectors of the extreme rays of polyhedral cone \mathcal{P} . We have the following fundamental theorem.

Theorem 7. *Let $\mathbf{R} \in \mathcal{R}_q$, there exists a $\mathcal{T} \subseteq \text{Extr}(\Gamma_N^q)$ such that $\mathbf{R} = \sum_{\mathbf{r}_i \in \mathcal{T}} \alpha_i \text{Proj}_{h_{X_k}, h_{U_e}, k \in [K], e \in E} \mathbf{r}_i$ with $\alpha_i \geq 0$.*

Proof: Follows from the the computation approach. ■

Before we show the construction of code to achieve an arbitrary point in the rate region, it is necessary to show that a rank function of \mathbb{F}_q -representable matroid M is associated with a linear network code in \mathbb{F}_q .

Given a particular MDCS problem, we first define a *network- \mathbb{F}_q -matroid mapping*, by loosening some conditions in [1], to be $f : X_{1:K} \cup \{U_e, e \in E\} \rightarrow \mathcal{S}'$, which associates each source variable and encoded variable with a collection of elements forming one set in a partition \mathcal{S}' of a ground set \mathcal{S} of a \mathbb{F}_q -representable matroid M , such that:

- 1.) f is a mapping from one element to one element in \mathcal{S}' ;
- 2.) $r(\bigcup_{k=1}^K f(X_k)) = \sum_{i=1}^K r(f(X_k))$,
- 3.) $r(f(\text{In}(v))) = r(f(\text{In}(v)) \cup f(\text{Out}(v))), \forall v \in E \cup D$, due to the encoder and decoder functions. Here, $\text{In}(v)$ is a collection of input variables to v and $\text{Out}(v)$ is a collection of output variables from v .

If all of the elements in \mathcal{S}' are singletons, then f is an one-to-one mapping to \mathcal{S} , and the matrix representation of M can be used as linear code in \mathbb{F}_q for this network, since the mapping guarantees the network constraints are obeyed. This coding solution is called a *basic scalar solution*. If \mathcal{S}' contains some elements that have cardinalities greater than 1, the representation of M is interpreted as a collection of bases of $|\mathcal{S}'|$ subspaces, which can also be used as a linear code and this solution is called a *basic vector solution*.

The basic solutions are in fact $(1, [r(f(\text{Out}(e))), e \in E])$ codes defined in the section III. In particular, there are $\sum_{k \in [K]} r(f(X_k))$ q -ary digits $X_{1:\sum_{k \in [K]} r(f(X_k))}$ (put \emptyset to position k if $H(X_k) = 0$). There exists a representation \mathbb{C} with dimension $(\sum_{k \in [K]} r(f(X_k))) \times (\sum_{k \in [K]} r(f(X_k)) + |E|)$ associated with the rank function r of M , where $\mathbb{C} = [\mathbb{I}_{\sum_{k \in [K]} r(f(X_k))} \mathbb{C}']$ and the identity matrix $\mathbb{I}_{\sum_{k \in [K]} r(f(X_k))}$ is mapped to the source digits and the rest \mathbb{C}' is mapped to coded messages such that $U_e = \mathbf{X} \mathbb{C}'_{(:, U_e)}, e \in E$, where $\mathbb{C}'_{(:, U_e)}$ indicates the columns mapped to message U_e ($\mathbb{C}'_{(:, U_e)} = \mathbb{O}_{\sum_{k \in [K]} r(f(X_k)) \times 1}$ if $H(U_e) = 0$). \mathbb{C}' can be a *semi-simplified solution* by deleting rows which are associated with sources with zero entropy but keeping the column size as $|E|$. In the following context, basic solutions are semi-simplified.

Now let us consider a point $\mathbf{R} \in \mathcal{R}_q$ associated

with source entropies $H_q(X_k), k \in [K]$. Suppose $\mathbf{R} = \sum_{\mathbf{r}_i \in \mathcal{T}} \alpha_i \text{Proj}_{h_{X_k}, h_{U_e}, k \in [K], e \in E} \mathbf{r}_i, \alpha_i \geq 0$, where $\mathcal{T} \subseteq \text{Extr}(\Gamma_N^q), \mathbf{r}_i \in \mathcal{T}$ there exists an associated semi-simplified basic scalar solution \mathbb{C}_i for the network.

Let $H_q(X_{k,i}), k \in [K]$ be the source entropies, $R_{e,i}, e \in E, i = 1, \dots, |\mathcal{T}|$ be the rates associated with \mathbf{r}_i . According to Theorem 7, $H(X_k) = \sum_{i=1}^{|\mathcal{T}|} \alpha_i H_q(X_{k,i})$ and $R_e = \sum_{i=1}^{|\mathcal{T}|} \alpha_i R_{e,i}$, where $H_q(X_{k,i}) \in \{0, 1\}$ and $R_{e,i} \in \{0, 1\}$ because they are from matroids.

The construction of a code to achieve \mathbf{R} is as follows.

- 1) Find rational numbers $\tilde{\alpha}_i = \frac{\tilde{t}_i}{n_i} \approx \alpha_i, i = 1, \dots, |\mathcal{T}|$, then $\tilde{H}(X_k) = \sum_{i=1}^{|\mathcal{T}|} \tilde{\alpha}_i H(X_{k,i})$ and $\tilde{R}_e = \sum_{i=1}^{|\mathcal{T}|} \tilde{\alpha}_i R_{e,i}$ are the approximation, which can be arbitrarily close, of source entropies and rates, respectively;
- 2) Let $L = \text{LCM}(\{n_i\})$ be the block length;
- 3) Suppose L blocks of all K source variables $X_{1:K}^{1:L}$ are losslessly converted to uniformly distributed q -ary digits by some fix-length source code using a sufficiently large number of outer blocks. We gather these q -ary digits formed by individually compressing the original source variables into a row vector $\tilde{\mathbf{X}}$, length $(\tilde{\mathbf{X}}) = L \sum_{k=1}^K \tilde{H}(X_k)$.
- 4) Let $\tilde{t}_i = L \tilde{\alpha}_i$ be the number of times we will use code \mathbb{C}'_i . For every time we use \mathbb{C}'_i , the number of q -ary digits encoded is equal to the number of rows in \mathbb{C}'_i (note that \mathbb{C}'_i is semi-simplified). So there exists a partition of $\tilde{\mathbf{X}}$ consisting of $\sum_{i=1}^{|\mathcal{T}|} \tilde{t}_i$ elements in total and all \tilde{t}_i elements mapped with \mathbb{C}'_i have the same cardinality which is the number of rows in $\mathbb{C}'_i, \forall i = 1, \dots, |\mathcal{T}|$. More specifically, we are drawing $\tilde{t}_i H(X_{k,i})$ samples from X_k 's buffer for the \tilde{t}_i repetitions of the basic solution \mathbb{C}'_i .
- 5) Let $\tilde{\mathbf{X}}' = \tilde{\mathbf{X}} \mathbb{G}$ (\mathbb{G} is a shuffled identity matrix to relocate the q -ary digits in $\tilde{\mathbf{X}}$) be a rearrangement of $\tilde{\mathbf{X}}$ such that the source digits are mapped in the same order as the basic solutions in the constructed code $\tilde{\mathbb{C}}$ which repeats \mathbb{C}'_i for \tilde{t}_i times, $\forall i \in \{1, 2, \dots, |\mathcal{T}|\}$ in the way as follows.

$$\tilde{\mathbf{U}} = \tilde{\mathbf{X}}' \times \text{BlkDiag}(\underbrace{\mathbb{C}'_1, \dots, \mathbb{C}'_1}_{\tilde{t}_1 \text{ times}}, \underbrace{\mathbb{C}'_i, \dots, \mathbb{C}'_i}_{\tilde{t}_i \text{ times}}, \underbrace{\mathbb{C}'_m, \dots, \mathbb{C}'_m}_{\tilde{t}_m \text{ times}}) \quad (5)$$

where $\text{BlkDiag}(\cdot)$ is a block diagonalizing function.

- 6) Note that all \mathbb{C}'_i have the same column size and the column indices are mapped to $e \in E$. Therefore, we can rearrange the columns in $\tilde{\mathbb{C}}$ to group all columns containing $\mathbb{C}'_{i,(:,U_e)}, i = 1, \dots, |\mathcal{T}|$ to be an encoding function for e . That is, $\mathbb{C} = \text{concatenation}(\mathbb{C}_{(:,U_e)}), \mathbb{C}_{(:,U_e)} = \tilde{\mathbb{C}}_{(:,(e+|E| \cdot [0: \sum_{i=1}^{|\mathcal{T}|} \tilde{t}_i - 1])})$. \mathbb{C} can be further simplified by deleting all-zero columns.

Indeed, we can see that the code constructed this way can achieve the point $\mathbf{R} \in \mathcal{R}_q$ by examining

$$\begin{aligned} H_q(\tilde{U}_e) &= \text{rank}(\mathbb{C}_{(:,U_e)}) = \sum_{i=1}^{|\mathcal{T}|} \tilde{t}_i \text{rank}(\mathbb{C}_{i,(:,U_e)}) = \sum_{i=1}^{|\mathcal{T}|} \tilde{t}_i R_{i,e} \\ &= L \sum_{i=1}^{|\mathcal{T}|} \tilde{\alpha}_i R_{i,e} = L \tilde{R}_e. \end{aligned}$$

Therefore, the actual rate per source variable is $\hat{R}_e = \frac{H_q(\tilde{U}_e)}{L} = \tilde{R}_e \approx R_e$, with arbitrarily small offset if the fraction

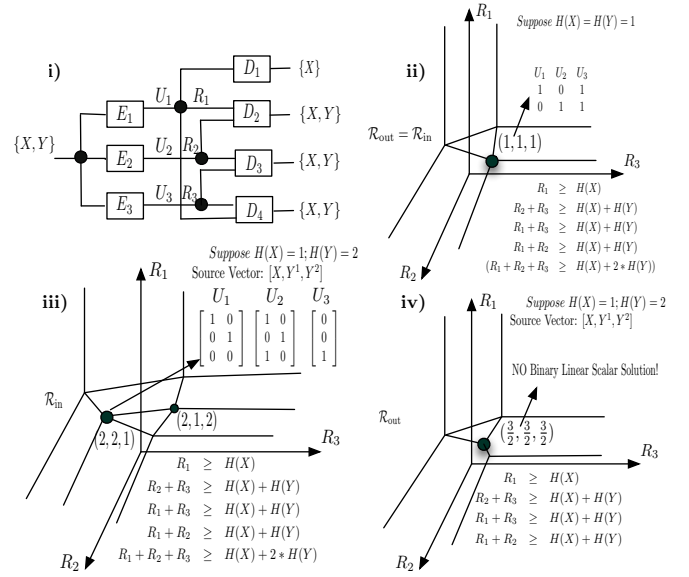


Figure 2. i) Example MDSC network. ii) Rate region and corresponding codes. iii) Scalar codes for inner bound. iv) No scalar code for outer bound.

approximations are arbitrarily close. If Γ_N^q is used in obtaining the rate region, $\mathbb{C}_i, \forall i = 1, \dots, |\mathcal{T}|$ are basic scalar solution, we call the constructed code a *scalar representation solution*. Similarly, if Γ_N^{space} is used in obtaining the rate region, some basic vector solutions \mathbb{C}_i , some $i = 1, \dots, |\mathcal{T}|$ may be needed in constructing the code \mathbb{C} . We call such a code involving basic vector solution(s) a *vector representation solution*.

Let's consider a 2-level-3-encoder MDSC as an example which is shown in Fig. 2. The outer \mathcal{R}_{out} and inner bound \mathcal{R}_{in} on rate region obtained from Shannon outer bound and binary inner bound are depicted in Fig. 2. Note that when $H(X) = H(Y)$, $\mathcal{R}_{\text{out}} = \mathcal{R}_{\text{in}}$, and an optimal code is provided in Fig. 2.

However, $\mathcal{R}_{\text{out}} \neq \mathcal{R}_{\text{in}}$ in general for this example, if $H(X) \neq H(Y)$. For $H(X) = 1, H(Y) = 2$ there is a gap between the inner and outer bounds. For the inner bound, we can find a scalar code solution. Fig. 2 shows a binary code to achieve the inner bound extreme point $\mathbf{R} = (2, 2, 1)$, which is a conic combination of two basic solutions

$$U_1^1 U_2^1 U_3^1 = Y^1 \times [1 \ 1 \ 0], \quad U_1^2 U_2^2 U_3^2 = XY^2 \times \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

For \mathcal{R}_{out} , we know there does not exist scalar binary coding solution for some extreme point. For example, as shown in Fig. 2, there is no scalar solution for the outer bound extreme point $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$. However, since we know the Γ_5^{space} makes up the gap and we know there must exist a solution to achieve this point. Actually, we can find a binary vector representation solution for this point. Note that we only need to group two outcomes of source variables and encode them together. Suppose we have source vector $v = [X_1, X_2, Y_1^1, Y_1^2, Y_2^1, Y_2^2]$ where the lower index indicates two outcomes in time while upper index indicates the position in one outcome. One vector representation coding solution is

$$U_1 U_2 U_3 = v \times \text{BlkDiag}(\mathbb{C}_1, \mathbb{C}_2) \quad (6)$$

which can also be expressed as a conic combination of two basic solutions

$$U_1^1 U_2^1 U_3^1 = X_1 Y_2^1 \times \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = X_1 Y_2^1 \times \mathbb{C}_1, \quad (7)$$

$$U_1^2 U_2^2 U_3^2 = \underbrace{X_2 Y_1^1 Y_1^2 Y_2^2}_w \times \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} = w \mathbb{C}_2.$$

VI. EXPERIMENTAL RESULTS

In this section, we review experimental results obtained with MATLAB implementations of the new computational approach and that of [9] on some MDCS problems, and provide new results about 3-level 3-encoder MDCS that could not previously be obtained.

A. Computation time comparison

Approach	2-level-3-encoder	3-level-3-encoder
Old in [9]	≥ 46 s	≥ 3600 s
New proposed	2.9 s	35 s

Both inner and outer bounds for rate regions were obtained for all 31 cases of the 2-level 3-encoder and all 69 cases of the 3-level 3-encoder MDCS problems enumerated in [6], using [9] and the new method in this paper. The average (over cases) execution time, measured in seconds for each rate region calculation, is displayed in the table above, indicate a significant speedup.

B. Numerical results for MDCS

The results presented in [9], included a full list of rate regions for 2-level 3-encoder MDCS proving that scalar binary codes are insufficient for 6 out of the 31 cases. Owing to the prohibitive complexity of [9] for 3-level-3-encoder MDCS, results for these larger problems were unobtainable using the technique in [9]. The method proposed in this paper makes 3-level-3-encoder MDCS and even more complicated networks quickly computable with simple MATLAB code. The results we obtained by calculating the 3-level 3-encoder MDCS regions are as follows.

There exist 15 out of 69 cases, where the rate region obtained from Shannon outer bound and binary representable matroid inner bound do not match, i.e we found gaps between these two bounds. The 15 cases include case numbers 8, 14, 28, 32, 37, 42, 47, 49, 53, 55, 57, 59, 63, 65, 69 from Hau's list [6].

One natural question is whether scalar linear codes over a larger field size and eliminate the gap in any of the 15 cases where scalar linear binary codes were insufficient. Our calculations showed that exactly the same achievable rate regions for both 2-level and 3-level MDCS problems with 3 encoders are obtained by considering the larger inner bound of matroids, i.e. by replacing Γ_N^{bin} with Γ_N^{mat} for $N \in \{5, 6\}$. That is, if there is some field size such that scalar linear codes over that field obtain the entire rate region then in all 100 cases that field size may be taken to be binary. This conclusion follows from the fact Γ_N^{mat} for $N \leq 7$ is an inner bound for $\bar{\Gamma}_N^*$, since

a result of Fournier showed that all matroids on ground set sizes ≤ 7 are representable [12].

These results demonstrated that simple scalar codes could not obtain the entire rate region for even these simple small MDCS networks. A natural alternative is to employ vector linear codes instead, which means encoding a group of outcomes of source variables for several time steps together. Passing from scalar codes to vector codes, in this sense, by replacing Γ_5^{bin} with Γ_5^{space} in our 2-level 3-encoder achievable rate regions, closes all of gaps, hence proving that the exact rate regions for 2-level-3-encoder MDCS are the same as that obtained from Shannon outer bound. This proves that vector linear codes (in the sense §V) suffice to obtain achieve all of the fundamental 2-level 3-encoder MDCS rate regions.

VII. CONCLUSION

This paper proposes a novel computational method inspired by double descriptions for computing inner and outer rate regions for MDCS and coded networks. The proposed double description manipulation of the extreme ray representation achieves significantly improved running times relative to a previous method utilizing Fourier-Motzkin projection, allowing several new results about small MDCS to be proven.

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