# Network Embedding Operations Preserving the Insufficiency of Linear Network Codes

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*Abstract*—**Operations for extending/embedding a smaller network into a larger network that preserve the insufficiency of classes of linear network codes are presented. Linear network codes over some finite field are said to be sufficient for a network if and only if for every point in the network coding rate region, there exists a code over that finite field to achieve it. Three operations are defined, and it is proven that they have the desired inheritance property, both for scalar linear network codes and for vector linear network codes, separately. Experimental results on the rate regions of multilevel diversity coding systems (MDCS), a sub-class of the broader family of multi-source multi-sink networks with special structure, are presented for demonstration. These results demonstrate that these notions of embedding operations enable one to investigate the existences of small numbers of forbidden network minors for sufficiency of linear network codes over a given field.**

## I. INTRODUCTION

When $\mathbb{F}_q$ linear codes suffice for a network, for every point in the network coding rate region of this network, there exists a $\mathbb{F}_q$ code or time-sharing between several $\mathbb{F}_q$ codes to achieve it. Due to the easy implementation and low computation complexity of linear network codes, especially simple linear network codes over small field sizes, resources can be saved if such a class of linear network codes suffice for a network. However, when a network is not small, it can be difficult to determine if a class of linear codes suffice or not.

This question is hard for general networks because of the following factors. First of all, it is difficult to obtain the exact rate region for a general multiterminal network, let alone to determine the codes to achieve it. Though it is known that the multiple source multicast network coding capacity region can be expressed in terms of the region of entropic vectors $\bar{\Gamma}_N^*$ [12], $\bar{\Gamma}_N^*$ is still unknown for $N \geq 4$. Secondly, though one can substitute in the outer and inner bounds on $\bar{\Gamma}_N^*$ to obtain outer and inner bounds on rate region and obtain the exact rate region by comparing them [6], [5] in some instances, good inner bounds are also difficult to obtain, especially as the size of the network $N$ grows. For instance, inner bounds based on ranks of linear subspaces are only known for $N \leq 5$ [2]. It is known that ranks of representable matroids can be used as linear space inner bounds on $\bar{\Gamma}_N^*$ [6], [5], [7], the enumeration of matroids is not an easy job, and currently the known list of all matroids is for $N \leq 9$ [9], [8]. While the techniques introduced in [1] enable one to directly enumerate classes of matroids, such as those that are extremal and representable over a certain field, or those obey the

constraints of a network, and thereby attack larger networks than $N = 9$, these techniques still suffer from an unavoidable fundamental combinatorial explosion in possibilities as the size of the network grows. This makes it difficult to determine the sufficiency or insufficiency of $\mathbb{F}_q$ linear codes through direct computation of the associated regions as the network size becomes large. This inspires us to search for methods of determining sufficiency/insufficiency of a class of codes other than direct computation of the regions.

It is known that, in matroid theory, if a matroid contains a forbidden minor that is not $\mathbb{F}_q$ representable, its extensions, together with itself, will not be $\mathbb{F}_q$ representable [10]. For instance, $U_{2,4}$ is not binary representable, and all matroids containing $U_{2,4}$ as a minor will not be binary representable. In fact, a matroid is binary representable if and only if it does not have $U_{2,4}$ as a minor, and the celebrated recently claimed proven Rota's conjecture states that there exists a finite list of forbidden minors for representability over any given field $\mathbb{F}_q$. If there exists similar property regarding the sufficiency of $\mathbb{F}_q$ codes for networks, one could easily determine that $\mathbb{F}_q$ codes do not suffice for a big network by knowing they do not suffice for one of its smaller "minor" networks.

In [4], a definition of *embedded* multilevel diversity coding system (MDCS) instances, a special subclass of general networks relevant for early models of distributed storage systems, was given. While this notion of embedding exclusively allowed the deletion of a source, it was observed that insufficiency of superposition (source separation) is preserved in the reverse of this operation. Inspired by this insightful idea, and by the notions of forbidden minors from matroid and graph theory, this paper defines three operations to obtain "minor" (embedded) networks, including source deletion, edge contraction and edge deletion. It is shown that, if $\mathbb{F}_q$ codes, vector or scalar, suffice for a big network, they will also suffice for the small network obtain by these operations. Equivalently, if $\mathbb{F}_q$ codes do not suffice for the small network, they will not suffice for the big network. A convenient byproduct of the analysis is an explicit relationship between the rate region of the larger network and the rate region of the smaller network.

To demonstrate the utility of the ideas, these operations and proven theorems are applied to multi-level diversity coding systems, a class of multi-source multi-sink networks. The experimental results demonstrate that thousands of distinct MDCS cases for which binary codes are insufficient can be grouped by considering only 12 forbidden minor networks,

which supports the power of investigating networks through the forbidden minor framework.

The remaining of this paper is organized as follows. We briefly review the network coding model and the rate region expressions in §II. In §III, the bounds on the region of entropic vectors, especially representable matroid bounds, are reviewed, followed by a discussion of their relationship with scalar and vector $\mathbb{F}_q$ codes to achieve them. The definitions of embedding operations on networks are given in §IV and the inheritance properties of insufficiency of a class of linear scalar/vector codes are given in §V. Finally, the supporting experimental results are presented in §VI and §VII concludes the paper.

## II. NETWORK MODEL AND RATE REGION

Suppose we have a network A represented as a directed acyclic graph, as shown in Figure 1. There are independent sources $\mathcal{S}$, communication links (edges) $\mathcal{E}$, intermediate nodes $\mathcal{G}$, and sink nodes $\mathcal{T}$. For each $s \in \mathcal{S}$, the associated source variable is $Y_s$ with source rate $\omega_s$. For an intermediate node $i \in \mathcal{G}$, we denote its incoming edges as $\text{In}(i)$ and outgoing edges as $\text{Out}(i)$. For each edge $e \in \mathcal{E}$, the associated random variable $U_e = f_e(\text{In}(i))$ is a function of all the input of node $i$, obeying the edge capacity $R_e$. The tail (head) node of edge $e$ is denoted as $\text{Tail}(e)$ ($\text{Head}(e)$). For each $t \in \mathcal{T}$, the output is $\beta(t) = g_t(\text{In}(t))$, is a subset of source variables and $\beta(t)$ can vary across $t$. Thus, a network A can be represented as the tuple $\mathsf{A} = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}, (\beta(t), t \in \mathcal{T}))$.

In [11], the source rate region for specific capacities $R_e, e \in \mathcal{E}$ is given in terms of $\bar{\Gamma}_N^*$. Note that source rates and edge capacities are pairs of tuples, there could be two types of problems. One is to fix edge capacities and investigate feasible sources rates. The other one is to fix source rates and investigate required edge capacities. To keep consistent with the experimental MDCS problems we will introduced in §VI, where edge capacities are investigated, we will give the expression of edge capacity region $\mathcal{R}(\mathsf{A})$ for the network A, with slight changes on the constraints and projection variables. If we collect all $Y_s, s \in \mathcal{S}$ and $U_e, e \in \mathcal{E}$ as random variables and assume $|\{Y_s \cup U_e, s \in \mathcal{S}, e \in \mathcal{E}\}| = N$, then

$$\mathcal{R}_{\omega_s}(\mathsf{A}) = \text{Ex}(\text{proj}_{U_e}(\text{con}(\Gamma_N^* \cap \mathcal{L}_{123}) \cap \mathcal{L}_{45})), \quad (1)$$

where $\Gamma_N^*$ is the region of entropic vectors (discussed in §III), $\text{con}(\mathcal{B})$ is the convex hull of $\mathcal{B}$, $\text{proj}_{U_e}(\mathcal{B})$ is the projection of the set $\mathcal{B}$ on the coordinates $(h_{U_e} | e \in \mathcal{E})$, and $\text{Ex}(\mathcal{B}) = \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : \mathbf{h} \geq \mathbf{h}' \text{ for some } \mathbf{h}' \in \mathcal{B}\}$, for $\mathcal{B} \subset \mathbb{R}_+^{2^N-1}$. Further, $\mathcal{L}_i, i = 1, 2, \ldots 5$ are network constraints representing source independency, source nodes coding, intermediate nodes coding, source rate constraints, sink nodes decoding, respectively (listed below). Finally, $\mathcal{L}_{123} = \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3$ and $\mathcal{L}_{45} = \mathcal{L}_4 \cap \mathcal{L}_5$. The network constraints naturally reflect the reproduction requirements and functional requirements of the network graph, namely:

$$\mathcal{L}_1 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_\mathcal{S}} = \Sigma_{s \in \mathcal{S}} h_{Y_s}\} \quad (2)$$

$$\mathcal{L}_2 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\text{Out}(k)}|Y_s} = 0\} \quad (3)$$

$$\mathcal{L}_3 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\text{Out}(i)}|X_{\text{In}(i)}} = 0\} \quad (4)$$

$$\mathcal{L}_4 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_s} \geq \omega_s\} \quad (5)$$

$$\mathcal{L}_5 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_{\beta(t)}|U_{\text{In}(t)}} = 0\}. \quad (6)$$
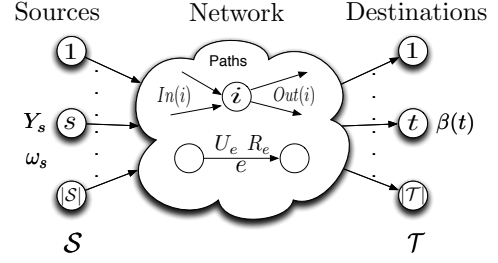


Figure 1: A general network model A

When the rate region of a network is computed in the most general sense, it is expressed as series of inequalities linking the lower bounds $\omega_s$ for a series of source entropies $H(X_s)$, and the link-capacity upper bounds $R_e$ for a series of link entropies $H(U_e)$. This is reflected in the equation (1), which is a function of some given source entropies $H(X_k), k \in \mathcal{S}$. However, to derive the capacity region of the network treating these entropies as variables, rather than fixed, this expression can be modified slightly by identifying the source rates $H(X_k) = \mathbf{h}_{Y_k}, k \in \mathcal{S}$ as long as $\mathbf{h} \in \Gamma_N^*$ is able to satisfy all the other constraints $\mathcal{L}_{1235}$ since this will guarantee that $\mathbf{h}$ also satisfies $\mathcal{L}_4$. Another issue in the computation is that we need to consider all exterior points after projecting on $h_{U_e}, e \in \mathcal{E}$. With a slight abuse of notation, if we define

$$\mathcal{L}_4' = \{(\mathbf{h}^T, \mathbf{R}^T)^T \in \mathbb{R}_+^{2^N-1+|\mathcal{E}|} : R_e \geq h_{U_e}, e \in \mathcal{E}\}, \quad (7)$$

the rate regions in (1), (9) and (10) which are expressed exclusively in terms of the variables $R_e, H(X_k), e \in \mathcal{E}, k \in \mathcal{S}$ will be

$$\mathcal{R}(\mathsf{A}) = \text{proj}_{R_\mathcal{E}, H(X_\mathcal{S})}(\overline{\text{con}(\Gamma_N^* \cap \mathcal{L}_{123})} \cap \mathcal{L}_5 \cap \mathcal{L}_4'), \quad (8)$$

where the projection on $H(X_\mathcal{S})$ is in fact projection on $\mathbf{h}_{Y_k}, k \in \mathcal{S}$ and on $R_\mathcal{E}$ is in fact on $R_e, e \in \mathcal{E}$. Note that, $\mathcal{L}_4'$ introduces new variables and is a cone in $2^N - 1 + |\mathcal{E}|$ dimensional space. In addition, since $\bar{\Gamma}_N^*, \mathcal{L}_{1,2,5}$ do not have dimensions on $R_e, e \in \mathcal{E}$, we add these free dimensions in the intersection. This alternate form (8) of (1) enables one to directly obtain a series of inequalities linking the link capacity variables $R_e$ with the source entropies $H(X_k)$.

While the analytical formation gives a possible way in principle to calculate the rate region of any network under network coding, we still have some problems. We know that $\Gamma_N^*$ is unknown and even not polyhedral for $N \geq 4$. Thus, the direct calculation of rate regions from equation (8) for a network with more than 4 variables is infeasible. However, replacing $\Gamma_N^*$ with polyhedral inner and outer bounds allows (8) to become a polyhedral computation problem which involves applying some constraints onto a polyhedra and then projecting down onto some coordinates. This inspires us to substitute $\Gamma_N^*$ with its closed polyhedral outer and inner bounds $\Gamma_N^{\text{out}}, \Gamma_N^{\text{in}}$ respectively, and get an outer and inner bound on the rate region:

$$\mathcal{R}_{\text{out}}(\mathsf{A}) = \text{proj}_{R_\mathcal{E}, H(X_\mathcal{S})}(\Gamma_N^{\text{out}} \cap \mathcal{L}_{1235} \cap \mathcal{L}_4'), \quad (9)$$

$$\mathcal{R}_{\text{in}}(\mathsf{A}) = \text{proj}_{R_\mathcal{E}, H(X_\mathcal{S})}(\Gamma_N^{\text{in}} \cap \mathcal{L}_{1235} \cap \mathcal{L}_4'). \quad (10)$$

If $\mathcal{R}_{\text{out}}(\mathsf{A}) = \mathcal{R}_{\text{in}}(\mathsf{A})$, we know $\mathcal{R}(\mathsf{A}) = \mathcal{R}_{\text{out}}(\mathsf{A}) = \mathcal{R}_{\text{in}}(\mathsf{A})$. Otherwise, tighter bounds are necessary.

In this work, we will follow (9) and (10) to calculate the rate region. Typically the Shannon outer bound $\Gamma_N$ and some inner bounds obtained from matroids, especially representable matroids, are used. We will briefly review the definition of these bounds in the next section, while details on the polyhedral computation methods with these bounds are available in [5], [6], [1].

## III. Construction of Bounds on Rate Region

Here we briefly review the definition of the region of entropic vectors $\Gamma_N^*$ and the polyhedral inner and outer bounds utilized for it to generate inner and outer bounds for network coding rate regions in this paper.

*1) Region of entropic vectors $\Gamma_N^*$:* Consider an arbitrary collection $\mathbf{X} = (X_1, \ldots, X_N)$ of $N$ discrete random variables with joint probability mass function $p_X$. To each of the $2^N - 1$ non-empty subsets of the collection of random variables, $X_{\mathcal{A}} := (X_i | i \in \mathcal{A})$ with $\mathcal{A} \subseteq \{1, \ldots, N\} \equiv [[N]]$, there is associated a joint Shannon entropy $H(X_{\mathcal{A}})$. Stacking these subset entropies for different subsets into a $2^N - 1$ dimensional vector we form an entropy vector

$$\mathbf{h} = [H(X_{\mathcal{A}}) | \mathcal{A} \subseteq [[N]], \mathcal{A} \neq \emptyset] \tag{11}$$

By virtue of having been created in this manner, the vector $\mathbf{h}$ must live in some subset of $\mathbb{R}_+^{2^N-1}$, and is said to be *entropic* due to the existence of $p_X$. However, not every point in $\mathbb{R}_+^{2^N-1}$ is entropic since for some many points, there does not exist associated valid distribution $p_X$. All entropic vectors form a region denoted as $\Gamma_N^*$. It is known that the closure of the region of entropic vectors $\bar{\Gamma}_N^*$ is a convex cone [11].

*2) Shannon outer bound $\Gamma_N$:* Next observe that elementary properties of Shannon entropies indicates that $H(X_{\mathcal{A}})$ is a non-decreasing submodular function, so that $\forall \mathcal{A} \subseteq \mathcal{B} \subseteq [[N]], \forall \mathcal{C}, \mathcal{D} \subseteq [[N]]$

$$H(X_{\mathcal{A}}) \leq H(X_{\mathcal{B}}) \tag{12}$$
$$H(X_{\mathcal{C} \cup \mathcal{D}}) + H(X_{\mathcal{C} \cap \mathcal{D}}) \leq H(X_{\mathcal{C}}) + H(X_{\mathcal{D}}). \tag{13}$$

Since they are true for any collection of subset entropies, these linear inequalities (12), (13) can be viewed as supporting halfspaces for $\Gamma_N^*$.

Thus, the intersection of all such inequalities form a polyhedral outer bound $\Gamma_N$ for $\Gamma_N^*$ and $\bar{\Gamma}_N^*$, where

$$\Gamma_N := \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} \left| \begin{array}{cc} h_{\mathcal{A}} \leq h_{\mathcal{B}} & \forall \mathcal{A} \subseteq \mathcal{B} \\ h_{\mathcal{C} \cup \mathcal{D}} + h_{\mathcal{C} \cap \mathcal{D}} \leq h_{\mathcal{C}} + h_{\mathcal{D}} & \forall \mathcal{C}, \mathcal{D} \end{array} \right. \right\}.$$

This outer bound $\Gamma_N$ is known as the *Shannon outer bound*, as it can be thought of as the set of all inequalities resulting from the positivity of Shannon's information measures among the random variables. While $\Gamma_2 = \Gamma_2^*$ and $\Gamma_3 = \bar{\Gamma}_3^*$, $\bar{\Gamma}_N^* \subsetneq \Gamma_N$ for all $N \geq 4$ [11], and indeed it is known [3] that $\bar{\Gamma}_N^*$ is not even polyhedral for $N \geq 4$.

*3) Matroid basics:* Matroid theory [10] is an abstract generalization of independence in the context of linear algebra to the more general setting of set systems. There are numerous equivalent definitions of matroids, we present the definition of matroids utilizing *rank functions*.

**Definition 1:** A set function $r_M : 2^{\mathcal{M}} \rightarrow \{0, \ldots, N\}$ is a rank function of a matroid if it obeys the following axioms:
1) Cardinality: $r_M(\mathcal{A}) \leq |\mathcal{A}|$;
2) Monotonicity: if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$ then $r_M(\mathcal{A}) \leq r_M(\mathcal{B})$;
3) Submodularity: if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ then $r_M(\mathcal{A} \cup \mathcal{B}) + r_M(\mathcal{A} \cap \mathcal{B}) \leq r_M(\mathcal{A}) + r_M(\mathcal{B})$.

Though there are many classes of matroids, we are especially interested in one of them, *representable matroids*, because they can be related to linear codes to solve network coding problems as discussed in [5], [6].

*4) Representable matroids:* Representable matroids are an important class of matroids which connect the independent sets to the notion of independence in a vector space.

**Definition 2:** A matroid $M$ with ground set $\mathcal{M}$ of size $|\mathcal{M}| = N$ and rank $r_M(\mathcal{M}) = r$ is representable over a field $\mathbb{F}$ if there exists a matrix $\mathbb{A} \in \mathbb{F}^{r \times N}$ such that for each independent set $I \in \mathcal{I}$ the corresponding columns in $\mathbb{A}$, viewed as vectors in $\mathbb{F}^r$, are linearly independent.

There has been significant effort towards characterizing the set of matroids that are representable over various field sizes, with a complete answer only available for fields of sizes two, three, and four. For example, a matroid $M$ is binary representable (representable over a binary field) iff it does not have the matroid $U_{2,4}$ as a minor. Here, a minor is obtained by series of operations of contraction and deletion [10]. $U_{k,N}$ is the *uniform* matroid on the ground set $\mathcal{M} = [[N]]$ with independent sets $\mathcal{I}$ equal to all subsets of $[[N]]$ of size at most $k$. For example, $U_{2,4}$ has as its independent sets

$$\mathcal{I} = \{\emptyset, 1, 2, 3, 4, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}. \tag{14}$$

Another important observation is that the first non-representable matroid is *Vámos* matroid, a well known matroid on ground set of size 8. That is to say, all matroids are representable, at least in some field, for $N \leq 7$.

*5) Inner bounds from representable matroids:* Suppose a matroid $M$ with ground set $\mathcal{M}$ of size $|\mathcal{M}| = N$ and rank $r_M(\mathcal{M}) = k$ is representable over the finite field $\mathbb{F}_q$ of size $q$ and the representing matrix is $\mathbb{A} \in \mathbb{F}_q^{k \times N}$ such that $\forall \mathcal{B} \subseteq \mathcal{M}$ $r_M(\mathcal{B}) = \text{rank}(\mathbb{A}_{:,\mathcal{B}})$, the matrix rank of the columns of $\mathbb{A}$ indexed by $\mathcal{B}$. Let $\Gamma_N^q$ be the conic hull of all rank functions of matroid with $N$ elements and representable in $\mathbb{F}_q$. This provides an inner bound $\Gamma_N^q \subseteq \bar{\Gamma}_N^*$, because any extremal rank function $\mathbf{r}$ of $\Gamma_N^q$ is by definition representable and hence is associated with a matrix representation $\mathbb{A} \in \mathbb{F}_q^{k \times N}$, from which we can create the random variables

$$(X_1, \ldots, X_N) = \mathbf{u}\mathbb{A}, \quad \mathbf{u} \sim \mathcal{U}(\mathbb{F}_q^k). \tag{15}$$

whose elements are $h_{\mathcal{A}} = r_M(\mathcal{A}) \log_2 q, \quad \forall \mathcal{A} \subseteq \mathcal{M}$. Hence, all extreme rays of $\Gamma_N^q$ are entropic, and $\Gamma_N^q \subseteq \bar{\Gamma}_N^*$. Further, if a vector in the rate region of a network is (projection of) a $\mathbb{F}_q$-representable matroid rank, the representation $\mathbb{A}$ can be used as a linear code to achieve that rate vector and this code is denoted as *basic scalar $\mathbb{F}_q$ code*. For an interior point in the rate region, which is the conic hull of projections of $\mathbb{F}_q$-representable matroid ranks, the code to achieve it can be

constructed by time-sharing between the basic scalar codes associated with the ranks involved in the conic combination. This code is denoted as *scalar $\mathbb{F}_q$ code*. Details on construction of such a code can be found in [5].

One can further generalize the relationship between representable matroids and entropic vectors established by (15) by partitioning the ground set $\mathcal{M} = \{1, \ldots, N'\}$ up into $N$ disjoint sets, $\mathcal{M}_1, \ldots, \mathcal{M}_N$ and defining for $n \in \{1, \ldots, N\}$ the new vector valued, random variables $\mathbf{X}'_n = [X_n | n \in \mathcal{M}_{n'}]$. The associated entropic vector will have entropies $h_A = r_M(\cup_{n \in A} \mathcal{M}_n) \log_2 q$, and is thus proportional to a *projection* of the original rank vector $\mathbf{r}$ keeping only those elements corresponding to all elements in a set in the partition appearing together. Thus, such a projection of $\Gamma^q_{N'}$ forms an inner bound to $\bar{\Gamma}^*_N$, which we will refer to as a *vector representable matroid inner bound* $\Gamma^q_{N,N'}$. As $N' \to \infty$, $\Gamma^q_{N,\infty}$ is the conic hull of all ranks of subspaces on $\mathbb{F}_q$. The union over all field sizes for $\Gamma^q_{N,\infty}$ is the conic hull of the set of ranks of subspaces. Similarly, if a vector in the rate region of a network is (projection of) a vector $\mathbb{F}_q$-representable matroid rank, the representation $\mathbb{A}$ can be used as a linear code to achieve that rate vector and this code is denoted as a *basic vector $\mathbb{F}_q$ code*. The time-sharing between such basic vector codes can achieve any point inside the rate region [5].

All the bounds discussed in this section could be used in equation (9) and (10) to calculate bounds on rate regions for a network A. If we substitute the Shannon outer bound $\Gamma_N$ into (9), we get

$$\mathcal{R}_{\text{out}}(\mathsf{A}) = \text{proj}_{R_{\mathcal{E}}, H(X_{\mathcal{S}})}(\Gamma_N \cap \mathcal{L}_{1235} \cap \mathcal{L}'_4). \qquad (16)$$

Similarly, when representable matroid inner bound $\Gamma^q_N$ and the vector representable matroid inner bound $\Gamma^q_{N,\infty}$ are substituted into (10), we get

$$\mathcal{R}_{s,q}(\mathsf{A}) = \text{proj}_{R_{\mathcal{E}}, H(X_{\mathcal{S}})}(\Gamma^q_N \cap \mathcal{L}_{1235} \cap \mathcal{L}'_4), \qquad (17)$$
$$\mathcal{R}_q(\mathsf{A}) = \text{proj}_{R_{\mathcal{E}}, H(X_{\mathcal{S}})}(\Gamma^q_{N,\infty} \cap \mathcal{L}_{1235} \cap \mathcal{L}'_4). \qquad (18)$$
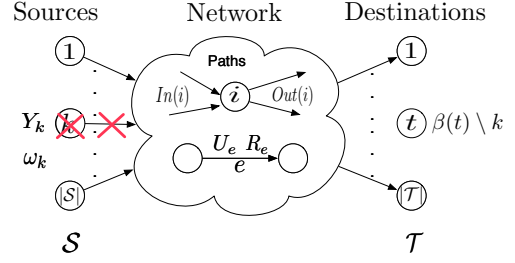
## IV. DEFINITION OF OPERATIONS

Here, we propose a series of embedding operations relating smaller networks with larger networks in a manner such that insufficiency of a class of linear network codes is inherited in the larger network from the smaller one.

The first operation is source deletion. When a source is deleted, the decoders that demand it will no longer demand it after deletion. In addition, the nodes and edges that only depend on this source will also be deleted.

**Definition 3** (Source Deletion)**:** Suppose a network $\mathsf{A} = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}, (\beta(t), t \in \mathcal{T}))$. When a source $k \in \mathcal{S}$ is deleted, denoted as $\mathsf{A} \backslash k$, in the new network $\mathsf{A}' = (\mathcal{S} \backslash k, \mathcal{G}', \mathcal{T}', \mathcal{E}', (\beta'(t), t \in \mathcal{T}'))$, we will have:
1) $\mathcal{G}' = \mathcal{G} \backslash \{i | \text{no path between } i, j, \forall j \in \mathcal{S} \backslash k\}$;
2) $\mathcal{T}' = \mathcal{T} \backslash \{t | \exists t' \in \mathcal{T}, t' \neq t, \beta(t) = \beta(t') \backslash k, \text{In}_{\mathsf{A}}(t') \subseteq \text{In}_{\mathsf{A}}(t)\}$;
3) $\mathcal{E}' = \{(s, i) | s \in \mathcal{S}', i \in \mathcal{G}'\} \cup \{(i, j) | i, j \in \mathcal{G}'\} \cup \{(i, t) | i \in \mathcal{G}', t \in \mathcal{T}'\}$.



(a) Demonstration of source deletion: when source $k$ is deleted, all nodes (and edges) that only depend on $k$, if any, will be deleted. Decoders that previously required $k$ will no longer require it.
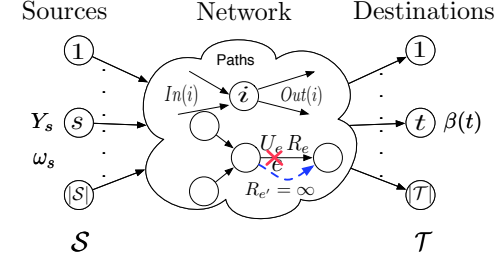
(b) Demonstration of edge contraction: when $e$ is contracted, its head node and tail node will be merged. Thus, the head node will directly have access to all tails of $\text{In}(\text{Tail}(e))$.
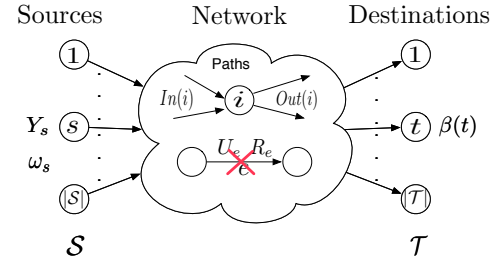
(c) Demonstration of edge deletion: when $e$ is deleted, its head node will no longer have access to its tail node.

Figure 2: Demonstration of operations on a network

4) $\forall t \in \mathcal{T}', \beta'(t) = \beta(t) \backslash k$;

Fig. 2(a) demonstrates the deletion of a source. When source $k$ is deleted, $t$ will no longer require $k$. The other edges and nodes that only depend on $k$, if any, will also be deleted.

Next, we consider the operation of contracting an edge. When an edge is contracted, the head node will directly have access to all the inputs of the tail node. This yields the same effective network as if the contracted edge had been replaced with a super edge with infinite capacity so that all the inputs of tail node could be directly sent to the head node.

**Definition 4** (Edge Contraction $(\mathsf{A}/e)$)**:** Suppose a network $\mathsf{A} = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}, (\beta(t), t \in \mathcal{T}))$. A smaller network $\mathsf{A}' = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}', (\beta(t), t \in \mathcal{T}))$ is obtained by contracting $e, e \in \mathcal{E}$, denoted by $\mathsf{A}/e$, if $\mathcal{E}' = (\mathcal{E} \backslash \{e\}) \cup \{e' = (\text{Tail}(e), \text{Head}(e)) | R_{e'} = \infty\}$.

Fig. 2(b) demonstrates the contraction of an edge. As it shows, when edge $e$ is contracted, the two nodes it connects will be connected with infinity capacity so that the head node of $e$ will directly have all the input of tail node of $e$. Next, we define edge deletion.

**Definition 5** (Edge Deletion (A\e))**:** Suppose a network $A = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}, (\beta(t), t \in \mathcal{T}))$. A smaller MDCS instance $A' = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}', (\beta(t), t \in \mathcal{T}))$ is obtained by deleting $e, e \in \mathcal{E}$, denoted by A\e, if $\mathcal{E}' = \mathcal{E} \setminus e$.

The essence of this definition is to keep the dependence relationship between input and output of its head node when an edge is deleted. In other words, when an edge is deleted, the head node should function as before deletion. Fig. 2(c) demonstrates the deletion of an edge. When edge $e$ is deleted, head node of $e$ no longer has access to $U_e$.

Next, we consider the order of different operations. It is not difficult to see that if a collection of sources are deleted, it does not matter which source is deleted first. Similarly, if a collection of edges are contracted or deleted, it does not matter which edge is operated first. Next we would like to show that different orders of operations give equivalent results.

**Theorem 1:** Let $\mathcal{O}_1, \mathcal{O}_2$ be two different operations on different elements, among the three operations defined in Definition 3– Definition 5. Applying $\mathcal{O}_1$ firstly and $\mathcal{O}_2$ secondly is equivalent to apply $\mathcal{O}_2$ firstly and $\mathcal{O}_1$ secondly.

*Proof:* We need to consider the 3 combinations.

**Edge deletion and edge contraction:** we need to show $(A\backslash e_1)/e_2 = (A/e_2)\backslash e_1$. We only need to consider the case when $e_1 \in \text{In}(\text{Tail}(e_2))$. If $e_1$ is deleted at first, when $e_2$ is contracted, the head node of $e_2$ will have access to tails of all the other edges that go into tail of $e_2$ except $e_1$. If $e_2$ is contracted first, $e_1$ will still not be available for head of $e_2$ since it is deleted.

**Edge deletion and source deletion:** we need to show $(A\backslash e_1)\backslash s = (A\backslash s)\backslash e_1$. We need to consider the case when $e_1$ only depends on $s$. In this case, no matter which operation is done first, $e_1$ and all other edges (nodes) that only depend on $s$ will be deleted.

**Edge contraction and source deletion:** we need to show $(A/e_1)\backslash s = (A\backslash s)/e_1$. We need to consider the case that $e_1$ only depends on $s$. In this case, $\text{In}(\text{Tail}(e_1))$ will also only depend on $s$. No matter which operation is done first, $e_1$, together with $\text{In}(\text{Tail}(e_1))$ will be deleted. The remaining network is the same. ∎

Based on these operations and Theorem 1, we can define an *embedded* network.

**Definition 6** (Embedded Network)**:** A network $A'$ is said *embedded* in another network A or a *minor* of A, denoted as $A' \prec A$, if $A'$ can be obtained by a series of operations of source deletion, edge deletion/ contraction on A. Reversely, we say that A is an extension of $A'$, $A \succ A'$.

## V. Preservation of Insufficiency of Linear Codes

Here we show that the property of the insufficiency of a class of linear network coding is inherited in the larger network from the smaller network under the embedding operations presented in the previous section.

**Theorem 2:** Suppose a network $A' = (\mathcal{S} \setminus k, \mathcal{G}', \mathcal{T}', \mathcal{E}', (\beta'(t), t \in \mathcal{T}'))$ is obtained by deleting $k$ from another network $A = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}, (\beta(t), t \in \mathcal{T}))$, then

$$\mathcal{R}(A') = \text{Proj}_{\mathbf{X}_{\setminus k}, \mathbf{R}_{\mathcal{E}'}} \left( \{ \mathbf{R} \in \mathcal{R}(A) \,|\, H(X_k) = 0 \} \right), \quad (19)$$

$$\mathcal{R}_q(A') = \text{Proj}_{\mathbf{X}_{\setminus k}, \mathbf{R}_{\mathcal{E}'}} \left( \{ \mathbf{R} \in \mathcal{R}_q(A) \,|\, H(X_k) = 0 \} \right), \quad (20)$$

$$\mathcal{R}_{s,q}(A') = \text{Proj}_{\mathbf{X}_{\setminus k}, \mathbf{R}_{\mathcal{E}'}} \left( \{ \mathbf{R} \in \mathcal{R}_{s,q}(A) \,|\, H(X_k) = 0 \} \right). \quad (21)$$

*Proof:* Select any point $\mathbf{R}' \in \mathcal{R}(A')$, then there exist random variables $\{\mathbf{X}_{\setminus k}, U_i, i \in \mathcal{E}'\}$ such that their entropies satisfy all the constraints in (1) determined by $A'$. Define $X_k$ to be the empty sources, $H(X_k) = 0$. Then the entropies of random variables $\{\mathbf{X}_{\setminus k}, U_i, i \in \mathcal{E}'\} \cup X_k$ will satisfy the constraints in A with $H(X_k) = 0$. Hence, the associated rate point $\mathbf{R} \in \{\mathbf{R} \in \mathcal{R}(A) | H(X_k) = 0\}$. Thus, we have $\mathcal{R}(A') \subseteq \text{Proj}_{\mathbf{X}_{\setminus k}, \mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}(A)|H(X_k) = 0\})$. If $\mathbf{R}'$ is achievable by $\mathbb{F}_q$ codes, since letting $X_k$ does not affect the other sources and codes, the same $\mathbb{F}_q$ code will also achieve the point $\mathbf{R}$ with $H(X_k) = 0$. Thus, $\mathcal{R}_q(A') \subseteq \text{Proj}_{\mathbf{X}_{\setminus s}, \mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(A)|H(X_k) = 0\})$ follows.

On the other hand, if we select any point $\mathbf{R} \in \{\mathbf{R} \in \mathcal{R}(A)|H(X_k) = 0\}$, we can see that $\mathbf{R}' = \text{Proj}_{\mathbf{X}_{\setminus k}, \mathbf{R}_{\mathcal{E}'}}(\mathbf{R}) \in \mathcal{R}(A')$ because $\mathbf{R}'$ is still entropic and the entropies of $\{\mathbf{X}_{\setminus k}, U_i, i \in \mathcal{E}'\}$ satisfy all constraints determined by $A'$. Thus, we have $\text{Proj}_{\mathbf{X}_{\setminus k}, \mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}(A)|H(X_k) = 0\}) \subseteq \mathcal{R}(A')$. If $\mathbf{R}$ is achievable by $\mathbb{F}_q$ code $\mathbb{C}$, then the code to achieve $\mathbf{R}'$ could be the code $\mathbb{C}$ with deletion of rows associated with source $X_k$, i.e., $\mathbb{C}' = \mathbb{C}_{\setminus X_k, :}$. Thus, $\text{Proj}_{\mathbf{X}_{\setminus k}, \mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(A)|H(X_k) = 0\}) \subseteq \mathcal{R}_q(A')$. ∎

**Theorem 3:** Suppose a network $A' = (\mathcal{S}, \mathcal{G}', \mathcal{T}, \mathcal{E}', (\beta(t), t \in \mathcal{T}))$ is obtained by contracting $e$ from another network $A = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}, (\beta(t), t \in \mathcal{T}))$, i.e., $A' = A \setminus e$, then

$$\mathcal{R}(A') = \text{Proj}_{H(X_k), k \in \mathcal{S}, \mathbf{R}_{\mathcal{E}'}} \mathcal{R}(A), \quad (22)$$

$$\mathcal{R}_q(A') = \text{Proj}_{H(X_k), k \in \mathcal{S}, \mathbf{R}_{\mathcal{E}'}} \mathcal{R}_q(A), \quad (23)$$

$$\mathcal{R}_{s,q}(A') \supseteq \text{Proj}_{H(X_k), k \in \mathcal{S}, \mathbf{R}_{\mathcal{E}'}} \mathcal{R}_{s,q}(A), \quad (24)$$

*Proof:* Select any point $\mathbf{R}' \in \mathcal{R}(A')$, then there exist random variables $\{\mathbf{X}_{\mathcal{S}}, U_i, i \in \mathcal{E}'\}$ such that their entropies satisfy all the constraints in (1) determined by $A'$. In the network A, define $U_e$ to be the concatenation of all input of tail node of $e$, $U_e = \mathbf{U}_{\text{In}(\text{Tail}(e))}$. Then the entropies of random variables $\{\mathbf{X}_{\mathcal{S}}, U_i, i \in \mathcal{E}'\} \cup U_e$ will satisfy the constraints in A, and additionally obey $H(U_e) = H(\mathbf{U}_{\text{In}(\text{Tail}(e))})$. Hence, the associated rate point $\mathbf{R} \in \left\{ \mathbf{R} \in \mathcal{R}(A) \,\middle|\, H(U_e) \geq H(\mathbf{U}_{\text{In}(\text{Tail}(e))}) \right\}$. Thus, we have

$$\mathcal{R}(A') \subseteq$$
$$\text{Proj}_{H(X_{\mathcal{S}}), \mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}(A)|H(U_e) \geq H(\mathbf{U}_{\text{In}(\text{Tail}(e))})\})$$
$$\subseteq \text{Proj}_{H(X_{\mathcal{S}}), \mathbf{R}_{\mathcal{E}'}} \mathcal{R}(A). \quad (25)$$

If $\mathbf{R}'$ is achievable by general $\mathbb{F}_q$ codes, since concatenation of all sources is a valid $\mathbb{F}_q$ code, we have

$$\mathcal{R}_q(A') \subseteq$$
$$\text{Proj}_{H(X_{\mathcal{S}}), \mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(A)|H(U_e) \geq H(\mathbf{U}_{\text{In}(\text{Tail}(e))})\})$$
$$\subseteq \text{Proj}_{H(X_{\mathcal{S}}), \mathbf{R}_{\mathcal{E}'}} \mathcal{R}_q(A). \quad (26)$$

However, we cannot establish same relationship when scalar $\mathbb{F}_q$ codes are considered, because for the point $\mathbf{R}'$, the associated $\mathbf{R}$ with $H(U_e)$ may not be scalar $\mathbb{F}_q$ achievable.

On the other hand, if we select any point $\mathbf{R} \in \mathcal{R}(\mathsf{A})$, we can see that $\mathbf{R}' = \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}\mathbf{R} \in \mathcal{R}(\mathsf{A}')$ because $\mathbf{R}'$ is still entropic and the entropies of $\{\mathbf{X}_{\mathcal{S}}, U_i, i \in \mathcal{E}'\}$ satisfy all constraints determined by $\mathsf{A}'$, since they are a subset of the constraints from $\mathsf{A}$. Thus, we have

$$\mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}\mathcal{R}(\mathsf{A}) \subseteq \mathcal{R}(\mathsf{A}'). \tag{27}$$

If $\mathbf{R} \in \mathcal{R}(\mathsf{A})$ is achievable by $\mathbb{F}_q$ code $\mathbb{C}$, either scalar or vector, then the code to achieve $\mathbf{R}' = \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}\mathbf{R} \in \mathcal{R}(\mathsf{A}')$ could be the code $\mathbb{C}$ with deletion of columns associated with encoder $E_e$, i.e., $\mathbb{C}' = \mathbb{C}_{:,\backslash E_e}$. Thus, we have

$$\mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}\mathcal{R}_q(\mathsf{A}) \subseteq \mathcal{R}_q(\mathsf{A}'), \tag{28}$$

$$\mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}\mathcal{R}_{s,q}(\mathsf{A}) \subseteq \mathcal{R}_{s,q}(\mathsf{A}'). \tag{29}$$

∎

**Theorem 4:** Suppose a MDCS instance $\mathsf{A}' = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}', (\beta(t), t \in \mathcal{T}))$ is obtained by deleting $e$ from another MDCS instance $\mathsf{A} = (\mathcal{S}, \mathcal{G}, \mathcal{T}, \mathcal{E}, (\beta(t), t \in \mathcal{T}))$, then

$$\mathcal{R}(\mathsf{A}') = \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(\mathsf{A}) | R_e = 0\}), \tag{30}$$

$$\mathcal{R}_q(\mathsf{A}') = \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(\mathsf{A}) | R_e = 0\}), \tag{31}$$

$$\mathcal{R}_{s,q}(\mathsf{A}') = \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_{s,q}(\mathsf{A}) | R_e = 0\}). \tag{32}$$

*Proof:* Select any point $\mathbf{R}' \in \mathcal{R}(\mathsf{A}')$, then there exist random variables $\{\mathbf{X}_{\mathcal{S}}, U_i, i \in \mathcal{E}'\}$ such that their entropies satisfy all the constraints in (1) determined by $\mathsf{A}'$. Let $U_e$ be empty set or encode all input with the all-zero vector, $U_e = \emptyset$. Then the entropies of random variables $\{\mathbf{X}_{\mathcal{S}}, U_i, i \in \mathcal{E}'\} \cup U_e$ will satisfy the constraints in $\mathsf{A}$, and additionally obey $H(U_e) \leq R_e = 0$. Hence, the associated rate point $\mathbf{R} \in \mathcal{R}(\mathsf{A}) \cap \{\mathbf{R} \in \mathcal{R}(\mathsf{A}) | R_e = 0\}$. Thus, we have

$$\mathcal{R}(\mathsf{A}') \subseteq \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}(\mathsf{A}) | R_e = 0\}). \tag{33}$$

If $\mathbf{R}'$ is achievable by general $\mathbb{F}_q$ linear vector or scalar codes, since all-zero code is a valid $\mathbb{F}_q$ vector and linear code, we have

$$\mathcal{R}_q(\mathsf{A}') \subseteq \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(\mathsf{A}) | R_e = 0\}), \tag{34}$$

$$\mathcal{R}_{s,q}(\mathsf{A}') \subseteq \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_{s,q}(\mathsf{A}) | R_e = 0\}). \tag{35}$$

On the other hand, if we select any point $\mathbf{R} \in \{\mathbf{R} \in \mathcal{R}(\mathsf{A}) | R_e = 0\}$, we can see that $\mathbf{R}' = \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\mathbf{R}) \in \mathcal{R}(\mathsf{A}')$ because $\mathbf{R}'$ is still entropic and the entropies of $\{\mathbf{X}_{\mathcal{S}}, U_i, i \in \mathcal{E}'\}$ satisfy all constraints determined by $\mathsf{A}'$. Thus, we have

$$\mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(\mathsf{A}) | R_e = 0\}) \subseteq \mathcal{R}(\mathsf{A}'). \tag{36}$$

If $\mathbf{R}$ is achievable by $\mathbb{F}_q$ code $\mathbb{C}$, no matter $\mathbb{C}$ is vector or scalar code, then the code to achieve $\mathbf{R}'$ could be the code $\mathbb{C}$ with deletion of columns associated with $U_e$, i.e., $\mathbb{C}' = \mathbb{C}_{:,\backslash U_e}$, because $U_e$ is sending nothing. Thus, we have

$$\mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(\mathsf{A}) | R_e = 0\}) \subseteq \mathcal{R}_q(\mathsf{A}'), \tag{37}$$

$$\mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}(\{\mathbf{R} \in \mathcal{R}_q(\mathsf{A}) | R_e = 0\}) \subseteq \mathcal{R}_{s,q}(\mathsf{A}'). \tag{38}$$

∎

**Corollary 1:** Given two networks $\mathsf{A}, \mathsf{A}'$ such that $\mathsf{A}' \prec \mathsf{A}$. If $\mathbb{F}_q$ linear codes suffice for $\mathsf{A}$, then $\mathbb{F}_q$ linear codes suffice for $\mathsf{A}'$. Equivalently, if $\mathbb{F}_q$ linear codes do not suffice for $\mathsf{A}'$, then $\mathbb{F}_q$ linear codes do not suffice for $\mathsf{A}$. Equivalently, if $\mathcal{R}_q(\mathsf{A}) = \mathcal{R}(\mathcal{A})$, then $\mathcal{R}_q(\mathcal{A}') = \mathcal{R}(\mathcal{A}')$.

*Proof:* From Definition 6 we know that $\mathsf{A}'$ is obtained by a series of operations of source deletion, edge deletion, edge contraction. Theorem 1 indicates that the order of the operations does not matter. Thus, it suffices to show that the statement holds when $\mathsf{A}'$ can be obtained by one of the operations on $\mathsf{A}$.

Suppose $\mathbb{F}_q$ linear codes suffice to achieve every point in $\mathcal{R}(\mathsf{A})$, i.e., $\mathcal{R}_q(\mathsf{A}) = \mathcal{R}(\mathsf{A})$. If $\mathsf{A}'$ is obtained by contracting one edge in $\mathsf{A}$, (22) and (23) in Theorem 3 indicate $\mathcal{R}_q(\mathsf{A}') = \mathcal{R}(\mathsf{A})$. Similarly, same conclusion is obtained from (19) and (20) in Theorem 2, (30) and (31) in Theorem 4, when $\mathsf{A}'$ is obtained by source deletion or deletion deletion. ∎

Note that scalar codes are a spacial class of general linear codes. If we only consider scalar linear codes, we will have the following similar corollary.

**Corollary 2:** Given two MDCS instances $\mathsf{A}, \mathsf{A}'$ such that $\mathsf{A}' \prec \mathsf{A}$. If $\mathbb{F}_q$ scalar linear codes suffice for $\mathsf{A}$, then $\mathbb{F}_q$ scalar linear codes suffice for $\mathsf{A}'$. Equivalently, if $\mathbb{F}_q$ scalar linear codes do not suffice for $\mathsf{A}'$, then $\mathbb{F}_q$ scalar linear codes do not suffice for $\mathsf{A}$. Equivalently, if $\mathcal{R}_{s,q}(\mathsf{A}) = \mathcal{R}(\mathcal{A})$, then $\mathcal{R}_{s,q}(\mathcal{A}') = \mathcal{R}(\mathcal{A}')$.

*Proof:* It follows a similar proof for Corollary 1. If $\mathcal{R}_{s,q}(\mathsf{A}) = \mathcal{R}(\mathcal{A})$, we can get $\mathcal{R}_{s,q}(\mathcal{A}') = \mathcal{R}(\mathcal{A}')$ from (30) and (32) in Theorem 4 or (19) and (21) in Theorem 2 if $\mathsf{A}'$ is obtained by deleting an encoder or source from $\mathsf{A}$.

Now we consider the case that $\mathsf{A}'$ is obtained by contracting an edge from $\mathsf{A}$. If $\mathcal{R}_{s,q}(\mathsf{A}) = \mathcal{R}(\mathcal{A})$, the projections $\mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}\mathcal{R}_{s,q}(\mathsf{A}) = \mathrm{Proj}_{H(X_{\mathcal{S}}),\mathbf{R}_{\mathcal{E}'}}\mathcal{R}(\mathcal{A})$. Together with (22) and (24), we will have $\mathcal{R}_{s,q}(\mathcal{A}') \supseteq \mathcal{R}(\mathcal{A}')$. It is trivial that $\mathcal{R}_{s,q}(\mathcal{A}') \subseteq \mathcal{R}(\mathcal{A}')$ because $\Gamma_N^q \subseteq \bar{\Gamma}_N^*$. Thus, we have $\mathcal{R}_{s,q}(\mathcal{A}') = \mathcal{R}(\mathcal{A}')$. ∎

## VI. Experimental Results on MDCS

Experiments are run on multilevel diversity coding system (MDCS) instances to support the theoretic results proven in previous section. For every MDCS instance, we calculate the outer and inner bounds on its rate region utilizing Shannon outer bound, representable matroid inner bounds. Sufficiency of scalar binary codes is investigated for 7382 MDCS instances. Before presenting the experimental results, we first give a brief introduction to MDCS.

A MDCS instance is a multi-source multi-sink network with special structure. In a MDCS instance, as shown in Fig. 3 and denoted as $\mathsf{A}$, there are $K$ *independent* sources $\mathbf{X}_{1:K} \equiv (X_1, \ldots, X_K)$ where source $k$ has support $\mathcal{X}_k$, and the sources are prioritized into $K$ levels with $X_1$ ($X_K$) the highest (lowest) priority source, respectively. As is standard in source coding, each source $X_k$ is in fact an i.i.d. sequence of random variables
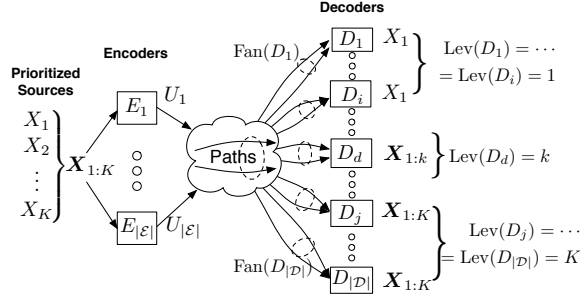
Figure 3: Diagram of a general MDCS instance A.

$\{X_k^t, t = 1, 2, \ldots\}$ in $t$, and $X_k$ is a representative random variable with this distribution.

All sources are made available to each of a collection of encoders indexed by the finite set $\mathcal{E}$. The output $\text{Out}(E_e)$ of an encoder $E_e$ is description/message variable $U_e, e \in \mathcal{E}$. The message variables are mapped to a collection of decoders indexed by the set $\mathcal{D}$ that are classified into $K$ levels, where a level $k$ decoder must losslessly (in the typical Shannon sense) recover source variables $\mathbf{X}_{1:k} = (X_1, \ldots, X_k)$, for each $k \in \{1, \ldots, K\}$. The mapping of encoders to decoders dictates the description variables that are available for each decoder in this recovery. The collection of mappings is denoted as a set $\mathcal{G}$, $\mathcal{G} \subseteq \mathcal{E} \times \mathcal{D}$ of edges where $(E_e, D_d) \in \mathcal{G}$ if $E_e$ is accessible by $D_d$. The set of encoders mapped to a particular decoder $D_d$ is called the *fan* of $D_d$, and is denoted as $\text{Fan}(D_d) = \{E_e | (E_e, D_d) \in \mathcal{G}\}$. Similarly, the set of decoders connected to a particular encoder $E_e$ is called the *fan* of $E_e$, and is denoted by $\text{Fan}(E_e) = \{D_d | (E_e, D_d) \in \mathcal{G}\}$. A decoder $D_d$ is said to be a level $k$ decoder, denoted by $\text{Lev}(D_d) = k$, if it wishes to recover exclusively the first $k$ sources $\mathbf{X}_{1:k}$. Different level $k$ decoders must recover the same subset of source variables using distinct subsets of description variables (encoders), among which one must not be subset of another. If we denote the input and output of a level $k$ decoder $D_d$ as $\text{In}(D_d)$ and $\text{Out}(D_d)$, respectively, we have $\text{In}(D_d) = \{U_e | E_e \in \text{Fan}(D_d), \forall e \in \mathcal{E}\}$ and $\text{Out}(D_d) = \mathbf{X}_{1:k}$.

We say that a MDCS instance is valid if it obeys the following constraints:

(**C1**.) If $\text{Lev}(D_i) = \text{Lev}(D_j)$, then $\text{Fan}(D_i) \not\subseteq \text{Fan}(D_j)$ and $\text{Fan}(D_j) \not\subseteq \text{Fan}(D_i)$;
(**C2**.) If $\text{Lev}(D_i) > \text{Lev}(D_j)$, then $\text{Fan}(D_i) \not\subseteq \text{Fan}(D_j)$;
(**C3**.) $\bigcup_{i \in \mathcal{D}} \text{Fan}(D_i) = \mathcal{E}$;
(**C4**.) There $\nexists k, l \in \mathcal{E}$ such that $\text{Fan}(E_k) = \text{Fan}(E_l)$.
(**C5**.) $\forall k \in \mathcal{S}, \exists d \in \mathcal{D}$ such that $\text{Lev}(D_d) = k$.

The first condition (**C1**) indicates that the fan of a decoder cannot be a subset of the fan of another decoder in the same level, for otherwise the decoder with access to more encoders would be redundant. The condition (**C2**) says that the fan of a higher level decoder cannot be a subset of the fan of a lower level decoder, for otherwise there exists a contradiction in their decoding capabilities. The condition (**C3**) requires that every encoder must be contained in the fan of at least one decoder. The condition (**C4**) requires that no two encoders have exactly
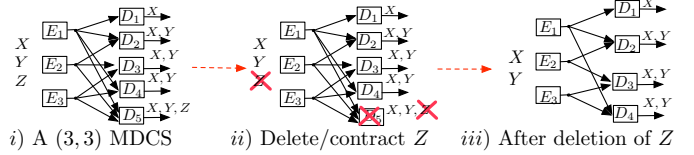


(a) Demonstration of source deletion: when source $Z$ is deleted, decoders that previously required it will no longer require it
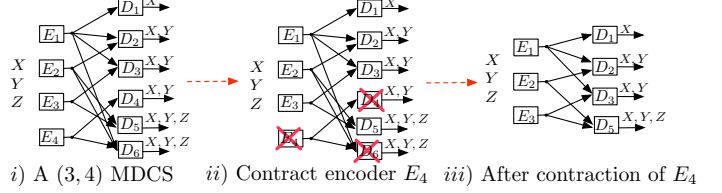


(b) Demonstration of encoder contraction: when $E_4$ is contracted, the fan of it will be deleted.
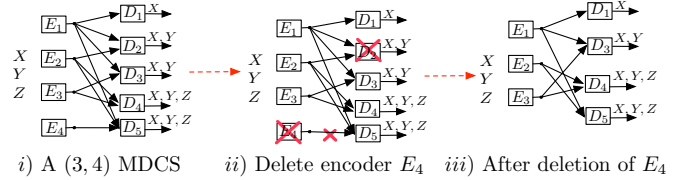


(c) Demonstration of encoder deletion: when $E_4$ is deleted, the fan of it will keep the same decoding abilities and lower-level decoders will be superseded if they have same fan as the fan of $E_4$ after deletion.

Figure 4: Demonstration of operations on MDCS instances.

the same fan, for otherwise the two encoders can be combined together. The condition (**C5**) ensures that there exists at least one decoder for every level.

In MDCS, encoders are the edges and there are no intermediate nodes. Therefore, the corresponding definitions of Definition 3– Definition 5 for MDCS are as follows.

**Definition 7** (Source Deletion $A \backslash X_k$)**:** Suppose a MDCS instance $A = (\{X_1, \ldots, X_K\}, \mathcal{E}, \mathcal{D}, \mathbf{L}', \mathcal{G}')$. When a source $X_k$ is deleted, denoted as $A \backslash X_k$, in the new MDCS instance $A' = (\{X_1, \ldots, X_K\} \backslash X_k, \mathcal{E}', \mathcal{D}', \mathbf{L}', \mathcal{G}')$, we will have:

1) $\mathcal{E}' = \mathcal{E}$;
2) $\mathcal{D}' = \mathcal{D} \backslash \{D_j | \exists D_i \in \mathcal{D}, i \neq j, \text{Out}_A(D_i) = \text{Out}_A(D_j) \backslash X_k, \text{Fan}_A(D_i) \subseteq \text{Fan}_A(D_j)$;
3) For $\mathbf{L}'$, $\text{Lev}_{A'}(D_d) = \text{Lev}_A(D_d) - 1, \forall D_d$ such that $X_k \in \text{Out}_A(D_d)$;
4) $\mathcal{G}' = \{(E_i, D_d) | E_i \in \mathcal{E}', D_d \in \mathcal{D}', (E_i, D_d) \in \mathcal{G}\}$.

This is straightforward because the deletion of a source just changes the decoding requirements of decoders. Fig. 4(a) demonstrates the deletion of a source. When source $Z$ is deleted, $D_5$ will no longer require $Z$ and thus becomes a level-2 decoder. However, since $D_2$ only has access to $E_1, E_2$ but is also a level-2 decoder, $D_5'$ becomes redundant and is deleted.

When an encoder is contracted, all of the decoders in its fan will be deleted, as well as all the edges associated with the contracted decoders, because the fan of this encoder will directly have access to all sources.

**Definition 8** (Encoder Contraction $(A/E_e)$)**:** Suppose a MDCS instance $A = (\{X_1, \ldots, X_K\}, \mathcal{E}, \mathcal{D}, \mathbf{L}, \mathcal{G})$. A smaller
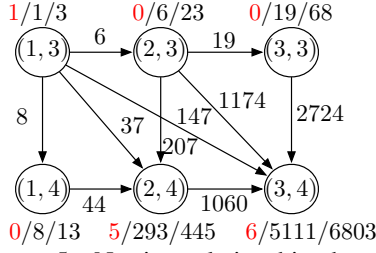
Figure 5: Nesting relationships between
$(1,3),(1,4),(2,3),(3,3),(2,4),(3,4)$ MDCS instances
regarding the sufficiency of scalar binary codes.

MDCS instance $\mathsf{A}' = (\{X_1,\ldots,X_K\},\mathcal{E}',\mathcal{D}',\mathbf{L}',\mathcal{G}')$ is obtained by contracting $E_e, e \in \mathcal{E}$, denoted by $\mathsf{A}/E_e$, if:

1) $\mathcal{E}' = \mathcal{E} \setminus E_e$;
2) $\mathcal{D}' = \mathcal{D} \setminus \{D_d | D_d \in \mathrm{Fan}_{\mathsf{A}}(E_e)\}$;
3) For $\mathbf{L}'$, $\mathrm{Lev}_{\mathsf{A}'}(D_i) = \mathrm{Lev}_{\mathsf{A}}(D_i), \forall D_i \in \mathcal{D}'$;
4) $\mathcal{G}' = \mathcal{G} \setminus \{(E_i,D_d) | E_i \in \mathcal{E}, D_d \in \mathrm{Fan}_{\mathsf{A}}(E_e), (E_i,D_d) \in \mathcal{G}\}$.

Fig. 4(b) demonstrates the contraction of an encoder. When encoder $E_4$ is contracted, all decoders which have access to $E_4$, i.e., fan of $E_4$, become redundant and are deleted.

**Definition 9** (Encoder Deletion $(\mathsf{A}\backslash E_e)$)**:** Suppose a MDCS instance $\mathsf{A} = (\{X_1,\ldots,X_K\},\mathcal{E},\mathcal{D},\mathbf{L},\mathcal{G})$. When an encoder $E_e \in \mathcal{E}$ is deleted, denoted as $\mathsf{A}\backslash E_e$, in the new MDCS instance $\mathsf{A}' = (\{X_1,\ldots,X_K\},\mathcal{E}',\mathcal{D}',\mathbf{L}',\mathcal{G}')$:

1) $\mathcal{E}' = \mathcal{E} \setminus E_e$;
2) $\mathcal{D}' = \mathcal{D} \setminus \{D_j \in \mathcal{D} | \exists D_i \in \mathrm{Fan}_{\mathsf{A}}(E_e) \setminus D_j, \mathrm{Fan}_{\mathsf{A}}(D_i) \setminus E_e \subseteq \mathrm{Fan}_{\mathsf{A}}(D_j)$ and $\mathrm{Lev}_{\mathsf{A}}(D_j) \le \mathrm{Lev}_{\mathsf{A}}(D_i)\}$;
3) For $\mathbf{L}'$, $\mathrm{Lev}_{\mathsf{A}'}(D_d) = \mathrm{Lev}_{\mathsf{A}}(D_d), \forall D_d \in \mathcal{D}'$;
4) $\mathcal{G}' = \{(E_i,D_d) | E_i \in \mathcal{E}', D_d \in \mathcal{D}', (E_i,D_d) \in \mathcal{G}\}$.

Fig. 4(c) demonstrates the deletion of an encoder. When encoder $E_4$ is deleted, $D_6$ no longer has access to $E_4$ but still has access to $E_1, E_2$. Note that, since level-2 decoder $D_2$ also has access to $E_1, E_2$, it becomes redundant and is deleted.

Fig. 5 shows our observations on the relationships between $(1,3),(1,4),(2,4)(2,3),(3,3),(3,4)$ MDCS instances regarding the sufficiency of scalar binary codes. We consider the non-isomorphic instances by removing symmetries of encoders. At each node, the first red numbers indicates the number of instances that do not have predecessors; the second number indicates the total number of scalar binary insufficient instances; the last number is the number of all non-isomorphic MDCS instances. The numbers on edges indicate the number of instances that have predecessors in the tail MDCS.

First, we observe that if scalar binary codes suffice for a MDCS instance, they will suffice for all its embedded instances as well. Furthermore, we observe that all 19 binary insufficient $(3,3)$ MDCS instances have minors of one of the 6 scalar binary insufficient $(2,3)$ MDCS instances, which themselves all have the same predecessor, the scalar binary insufficient $(1,3)$ MDCS instance, same as all the 8 scalar binary insufficient $(1,4)$ MDCS instances. However, for $(2,4)$ $((3,4))$ MDCS, there are 5 (6) instances that we cannot find predecessors for them using these three operations. The

numbers on every edge indicates how many scalar binary insufficient head MDCS instances have predecessors in the tail MDCS instances. Therefore, we have a list of 12 forbidden minor networks, which may not be complete, for scalar binary codes to suffice for MDCS instances.

## VII. CONCLUSION

This paper presented a series of tools to enable a systematic study of classes of networks for which scalar/vector linear network codes over a given field are not capable of achieving all possible rates in the network coding rate region. In particular, three embedding operations relating larger networks with smaller networks were presented. It was proven that if scalar/vector linear codes over a given field were sufficient to achieve all of the points in the rate region for the larger network, they would also be sufficient to achieve all of the points in the rate region for the smaller network. This enables one to investigate the sufficiency of a class of codes via a list of forbidden minor embedded networks under these operations. The utility of the technique was demonstrated in the context of MDCS networks, in which it was shown that the thousands of small non-isomorphic MDCS networks for which scalar binary codes are insufficient can all be boiled down to the inclusion of just 12 forbidden MDCS minor networks. Drawing another analogy to matroid theory, an important direction for future research is to determine if this list of forbidden minor networks under these operations will be finite for given fields.

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