Relationships Among Bounds for the Region of Entropic Vectors in Four Variables

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***Abstract*—Extreme point representations are utilized together with facet membership to gain insights about the relationship between various bounds on the set of normalized entropic vectors in 4 random variables. Echoing prior results from Matúš (1995), we begin by showing that the 206 extreme points of the Shannon outer bound comprise 33 forms where 16 of these forms are entropic and achievable with binary random variables (one bit), 16 of the forms are entropic and achievable using two bits, and the remaining single form of six extreme points is not entropic, not achievable with two bits, and is the only violator of the Ingleton inequality. Thus, the Ingleton inner bound is matched to the Shannon outer bound in that it contains all entropic Shannon extreme points. Further novel investigation of the structure of the Ingleton and Shannon polytopes reveal that *every* facet of the Shannon outer bound requires two bits per variable and can be improved upon. The inner bound formed from the convex hull of the set of binary entropic vectors is shown to neither contain nor be contained in the Ingleton inner bound, and is shown to not be a polytope. The addition of the non-Shannon-type inequality due to Zhang and Yueng plus the six non-Shannon-type inequalities due to Dougherty, Freiling, and Zeger yield a polytopic outer bound where none of the non-Shannon extreme points obey Ingleton or are binary achievable. Worst case distance gaps between the various bounds are calculated.**

I. Introduction

As all rate regions in multi-terminal information theory may be interpreted as linear projections of the set of entropic vectors under appropriate distribution constraints, determining accurate bounds for the region of entropic vectors is of fundamental importance. While the region of entropic vectors in two and three variables is exhaustively characterized by Shannon's inequalities reflecting the positivity of information measures, Zhang and Yeung proved that other, non-Shannon, inequalities are necessary for $N \geq 4$ random variables [1], [2]. Despite promising connections with group theory and an associated representation of the entropic region involving a convex hull of an unbounded number of vectors formed from finite groups [3], [4], [5], a computable representation of the region of entropy vectors is still unavailable even for four variables. Significant insight into the difficulty of the problem of providing such a characterization has recently been gained by a result from Matúš showing that the region of entropic vectors is not polyhedral for $N \geq 4$ [6].

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Given a joint distribution on $N \geq 2$ discrete random variables, the entropy vector associated with that distribution is the $2^N - 1$-dimension vector with an entry for the Shannon entropy associated with the joint distribution on each non-empty subset of those random variables. The set of entropic vectors, denoted Γ_N^*, is a region in $\mathbb{R}^{2^N-1}$ containing those points for which there exists a joint distribution on N random variables with those entropies. The closure of the set of entropy vectors, denoted $\bar{\Gamma}_N^*$, is well-known to be a convex cone [7]. As mentioned above, more recently this cone has been established to be non-polyhedral, i.e., the surface is curved, for $N \geq 4$ [6].

A. Focus: extreme points, binary achievable, inner bounds

Our focus in this paper has three aspects that merit emphasis: $i)$ we focus on the extreme point representation of the polytopic outer bounds to $\bar{\Gamma}_N^*$ instead of the more common focus on the intersection of half-spaces, $ii)$ we give special attention to the set of binary achievable entropy vectors, and $iii)$ we focus on inner bounds to $\bar{\Gamma}_N^*$ instead of on outer bounds. We emphasize that other papers also focus on these three things, but these three aspects are less common in the technical literature. A summary of the specific contributions of our work is given later.

Extreme points. Any polyhedra admits two dual representations: as a set of linear inequalities specifying an intersection of half-spaces, and as the convex hull of a set of extreme points and rays. Much of the literature about the (closure of) the set of entropic vectors $\bar{\Gamma}_N^*$ and bounds for it has focused on representations of these convex sets as an intersection of half-spaces. Such an interest is natural given that the context of studying $\bar{\Gamma}_N^*$ has often been to study linear information inequalities, and these are exactly supporting half-spaces of this set. In this paper, we study the alternate extreme point representation of these sets[1]

Binary achievable. Our focus is on the set of binary achievable entropy vectors. An entropy vector is binary achievable if there exists a joint distribution on N *binary* random variables (i.e., N bits) with the corresponding entropies. Binary entropic vectors are "simplest" in that the size of their support is as small as possible. This simplicity motivates

[1] Matúš (1995) [8], [9] also considers the extreme point representation.

their study, since it is not known what portion of $\bar{\Gamma}_N^*$ is achievable using just bits. In fact we focus on the *convex hull* of the set of binary achievable entropy vectors, denoted $\text{conv}(\Phi_N)$. We further study the set of two-bit achievable entropy vectors, where "two-bit" refers to using N random variables, each taking at most four values (hence two bits). Clearly binary achievable (one bit) entropic vectors are also two bit achievable, but we use the phrase "two bit achievable" to implicitly refer to those entropy vectors that are two bit achievable but not one bit achievable.

Inner bounds. Significant attention has been given to the use of non-Shannon-type information inequalities for obtaining outer bounds on $\bar{\Gamma}_N^*$ that improve upon the polytopic outer bound formed by the Shannon inequalities, e.g., [1], [2], [10]. Our focus in this paper is on inner bounds. The best known inner bound is the set of entropy vectors that obey the Ingleton inequality [8], [2]. Clearly the set $\text{conv}(\Phi_N)$ also forms an inner bound. We study the relationship between these two inner bounds in this paper.

Matúš [8] also studies the relationship between the Shannon outer bound and the Ingleton inner bound. In particular he $i)$ characterizes the Shannon outer bound in terms of a set of extreme rays, $ii)$ provides a "minimal" probabilistic construction for each extreme ray in the Ingleton inner bound, and $iii)$ identifies those extreme rays that are common and distinct between the Shannon and Ingleton constructions[2].

Our investigations presented in this paper, summarized below, are complementary to these findings.

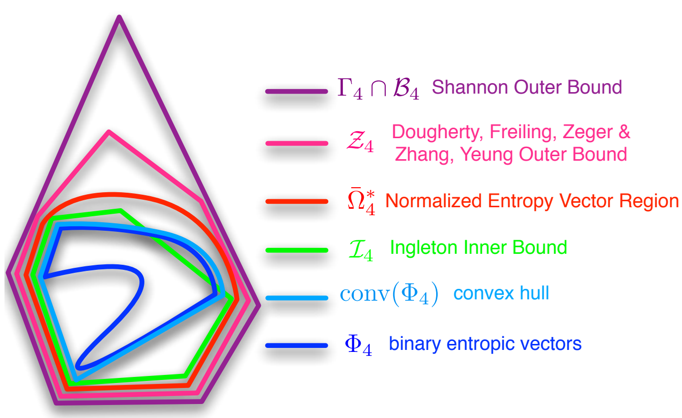

Fig. 1. Illustration of the relationships among the various bounds on the normalized entropy vector region $\bar{\Omega}_4^*$ (red). The normalized Shannon outer bound $\Gamma_4 \cap \mathcal{B}_4$ (purple) encloses the tighter outer bound $\mathcal{Z}_4$ (pink) provided by adding the non-Shannon type inequalities of Dougherty, Freiling and Zeger [10] and Zhang and Yeung [1]. The Ingleton inner bound $\mathcal{I}_4$ (green) and the convex hull of the set of binary achievable entropy vectors $\text{conv}(\Phi_4)$ (light blue) are complementary inner bounds in that neither contains the other, although $\mathcal{I}_4$ has the virtue of being a polytope while $\text{conv}(\Phi_4)$ is not.

[2]The focus in [8] is on conditional dependence relationships, and the connection with entropic vectors and the Shannon and Ingleton bounds is not always explicit.

B. Summary of contributions

Specific novel contributions of this paper include:

Shannon outer bound. Every Shannon outer bound facet contains a extreme point that is non-entropic and Ingleton-violating. Further, every Shannon outer bound facet has a (full 14 dimensional) subset that is binary achievable, and the remaining entropic extreme points can be achieved with two bits. That is the Shannon outer bound is loose on every facet (since it has a non-entropic extreme extreme point), but is also partially tight on every facet (since each facet has at least 14 linearly independent binary achievable extreme extreme point).

Ingleton and binary inner bounds. The Ingleton inner bound does not contain the set of binary entropic vectors, and the set of binary entropic vectors does not contain the Ingleton inner bound. That is, the two inner bounds are complementary. In particular, we provide explicit constructions of entropy vectors that $i)$ violate Ingleton but which are binary entropic, and $ii)$ satisfy Ingleton but are not binary achievable. In fact, we further show that the Ingleton inner bound is strictly contained within the convex hull of the set of 3-bit entropic vectors (i.e. entropy vectors among 4 variables formed from three bits each).

Bound tightness and looseness. Combining the above two observations, we conclude that both the Ingleton inner bound and the binary achievable inner bound are both tight and loose on each facet of the Shannon outer bound. They are both tight in that each inner bound has a full-dimensional intersection with each Shannon outer bound facet, but they are each loose in that each Shannon outer bound facet contains a non-entropic extreme extreme point. Further, we show that the percentage of faces of the Shannon outer bound of dimension d with all entropic extreme points (i.e., no Ingleton violating non-entropic extreme points) decreases in d (from 97% of the extreme points ($d = 0$) being entropic to 0% of the facets ($d = 14$) having a non-entropic extreme extreme point).

The DFZ outer bound. An improved outer bound (relative to the Shannon outer bound) is obtained by adding the six new forms of non-Shannon-type information inequalities by Dougherty, Freiling, and Zeger [10] and the information inequality due to Zhang and Yeung [2]. This new polytopic outer bound has an additional 120 extreme points not found in the Shannon outer bound. None of these new extreme points is binary achievable and all of them violate the Ingleton inequality. Hence, using these techniques, it is not yet possible to find any new extreme points of the set of normalized entropic vectors among the extreme points of the DFZ outer bound.

Speaking loosely, the uniting theme of these findings is that there is a very intricate and delicate interconnection among the Shannon outer bound, the Ingleton inner bound, and the binary achievable inner bound. Some of these findings are summarized in Fig. 1.

The rest of this paper is organized as follows: §II discusses tuned inner and outer bounds on the entropy region, §III classifies the extreme points and facets of the Shannon outer bound and the improved outer bound due to [2], [10], §IV computes the worst-case gap between the inner and outer bounds, and §V concludes the paper.

II. Entropy region tuned inner/outer bounds

Every compact convex set has two representations: one as an intersection of its supporting half-spaces, and the other as the convex hull of its extreme points. An extreme point of a convex set is a point which can not be be expressed as a convex combination of any points in $\mathcal{C}$ other than itself. When the compact convex set is a polytope, both the number of half-spaces and the number of extreme points are taken finite. When collected together, these dual representations often allow concepts that are difficult to discern in only one of the representations alone to be elucidated using the other.

The extreme point representation concept can be especially useful when attempting to bound an unwieldy compact convex set which does not admit a simple representation. This is because any extreme point $\mathbf{x}$ of a convex outer bound $\mathcal{O}$ to the unwieldy convex set $\mathcal{C}$ ($\mathcal{C} \subseteq \mathcal{O}$) which itself lies in $\mathcal{C}$ (i.e., has $\mathbf{x} \in \mathcal{C}$) must *also* be an extreme point of $\mathcal{C}$.

This fact inspires a technique for producing an inner bound $\mathcal{I}$ for the unwieldy set $\mathcal{C}$ tuned to any given polytopic outer bound $\mathcal{O}$. In particular, one first determines the dual representation of the outer bound, using, e.g., the program lrs [11], yielding a list of its extreme points. Next, one checks each of these extreme points for membership in $\mathcal{C}$. Those extreme points of $\mathcal{O}$ which also lie in $\mathcal{C}$ are then collected into a set $\mathcal{V}$, and the inner bound $\mathcal{I} \subset \mathcal{C}$ is formed as the convex hull of these points $\mathcal{I} = \mathrm{conv}(\mathcal{V})$. The dual representation of $\mathcal{I}$ in terms of half-spaces can, of course, be computed from its extreme points of $\mathcal{V}$, again, e.g., using an implementation of the double description method or the program lrs [11]. The inner bound $\mathcal{I}$ has the benefit of being formed exclusively from extreme points of the unwieldy convex set, and thus will yield the same answer as the outer bound $\mathcal{O}$ under a linear program with optimal solution anywhere where $\mathcal{O}$ is tight on the unwieldy set $\mathcal{C}$. This methodology is depicted in Fig. 2.

In this paper, the unwieldy set of interest is the convex cone which is the closure of the set of entropic vectors $\bar{\Gamma}^*_N$ for $N \geq 4$. In order to make this set bounded (and hence compact) we will actually work with the intersection $\bar{\Gamma}^*_N \cap \mathcal{B}_N$, where

$$\mathcal{B}_N = \{\mathbf{h} \in \mathbf{R}^{2^N-1} | H(X_i) \leq 1, i \in \{1, \dots, N\}\} \qquad (1)$$

is the unit box in $\mathbf{R}^{2^N-1}$. This set is unwieldy in the sense that we have computable concise outer bounds for it, but a computable[3] indicator function for membership in it is still unavailable. This may seem to make it impossible to apply the inner bounding technique, since determining membership in $\bar{\Gamma}^*_N \cap \mathcal{B}_N$ is the hard problem to begin with.

However, our earlier work has developed a computable indicator function for testing whether or not a candidate vector is an entropy vector for N *binary* random variables [12]. That is, we have developed a computable algorithm for testing membership in the set of binary entropy vectors, Φ_N, but there is no known computable algorithm for testing membership in $\bar{\Gamma}^*_N$. In fact [12] also establishes that all extreme points of $\bar{\Gamma}^*_N$ are binary for $N = 2, 3$.

[3] We say a function is computable if we have a finite terminating algorithm to evaluate that function.

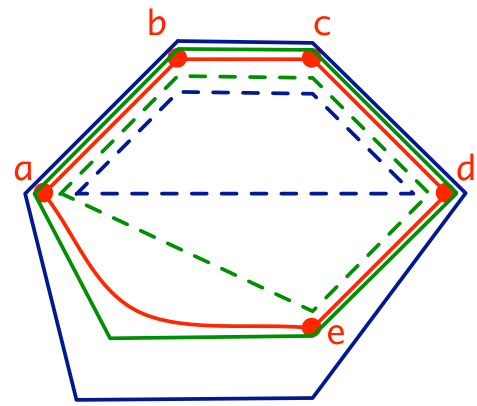

Fig. 2. Illustration of the general principle of improved outer bounds yielding improved inner bounds. The convex set in red has extreme extreme points a,b,c,d,e and a curved face between a,e. A loose outer bound (blue solid lines) has extreme points a,b,c,d which yields the loose inner bound formed by the convex hull of a,b,c,d (blue dotted lines). An improved outer bound (green solid lines) achieves an additional extreme point e, and therefore yields an improved corresponding inner bound, the convex hull of a,b,c,d,e (green dotted lines). Roughly speaking, the correspondence with the entropy vector region is as follows. The "coarse" Shannon outer bound yields a matched Ingleton inner bound. A "refined" outer bound using the Zhang and Yeung, and Dougherty, Freiling, and Zeger non-Shannon-type inequalities may yield additional entropic extreme points that would produce an inner bound that improves upon Ingleton.

III. Classification of the extreme points and facets of two outer bounds

A. *The Shannon Outer Bound* Γ_4

The Shannon outer bound Γ_N is a convex polyhedral cone containing the convex non-polyhedral cone $\bar{\Gamma}^*_N$. We prefer to work instead with bounded convex sets obtained by intersection with $\mathcal{B}_N$ defined in (1). In particular, $\Gamma_N \cap \mathcal{B}_N$ is the normalized Shannon outer bound, a bounded compact convex polytope and contains the normalized set of entropy vectors $\bar{\Gamma}^*_N \cap \mathcal{B}_N$, a bounded compact convex non-polytopic set.

Focusing on the specific case of $N = 4$, the minimal set of Shannon information inequalities that characterize the set $\Gamma_N \cap \mathcal{B}_N$ as the intersection of supporting half-spaces contain 32 inequalities found in five distinct forms. The number of permutations of each inequality in the minimal representation of the Shannon outer bound are listed next to each inequality.

inequality form	# in form
$1 - h_i \geq 0$	4
$h_k + h_l - h_{kl} \geq 0$	6
$-h_i + h_{ik} + h_{il} - h_{ikl} \geq 0$	12
$-h_{ij} + h_{ijk} + h_{ijl} - h_{ijkl} \geq 0$	6
$-h_{ijk} + h_{ijkl} \geq 0$	4

The extreme points of the normalized Shannon outer bound consist of 206 extreme points, which are random variable

#	h_1	h_2	h_3	h_4	h_{12}	h_{13}	h_{14}	h_{23}	h_{24}	h_{34}	h_{123}	h_{124}	h_{134}	h_{234}	h_{1234}	X_1	X_2	X_3	X_4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	b_1	b_1	b_1	b_1
4	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	b_1	b_1	b_1
4	1	1	1	1	2	2	2	1	1	1	2	2	2	1	2	b_1	b_2	b_2	b_2
6	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	b_1	b_1	b_2	$b_1 \oplus b_2$
6	0	1	0	1	1	0	1	1	1	1	1	1	1	1	1	0	b_1	0	b_1
12	0	1	1	1	1	1	1	2	1	2	2	1	2	2	2	0	b_1	b_2	b_1
6	1	1	1	1	2	2	2	2	1	2	3	2	3	2	3	b_1	b_2	b_3	b_2
4	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	0	0	0	b_1
6	0	0	1	1	0	1	1	1	1	2	1	1	2	2	2	0	0	b_1	b_2
4	0	1	1	1	1	1	1	2	2	2	2	2	2	3	3	0	b_1	b_2	b_3
1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	4	b_1	b_2	b_3	b_4
4	0	1	1	1	1	1	1	2	2	2	2	2	2	2	2	0	b_1	b_2	$b_1 \oplus b_2$
4	1	1	1	1	2	2	2	2	2	2	3	3	3	2	3	b_1	b_2	b_3	$b_2 \oplus b_3$
3	1	1	1	1	2	2	1	1	2	2	2	2	2	2	2	b_1	b_2	b_2	b_1
1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	b_1	b_2	b_3	$b_1 \oplus b_2 \oplus b_3$

TABLE I

THE FORMS OF SHANNON OUTER BOUND EXTREME POINTS THAT ARE BINARY ACHIEVABLE, AND EXAMPLE CONSTRUCTIONS FOR THEM FROM 4 UNIFORMLY DISTRIBUTED INDEPENDENT BITS b_1, b_2, b_3, b_4.

#	h_1	h_2	h_3	h_4	h_{12}	h_{13}	h_{14}	h_{23}	h_{24}	h_{34}	h_{123}	h_{124}	h_{134}	h_{234}	h_{1234}	X_1	X_2	X_3	X_4
4	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	$(b_1, 0)$	$(b_2, 0)$	$(b_1 \oplus b_2, 0)$	(b_1, b_2)
12	1	1	2	2	2	3	2	3	2	3	3	2	3	3	3	$(b_1, 0)$	$(b_2, 0)$	$(b_3, b_1 \oplus b_2)$	(b_1, b_2)
12	1	2	2	2	3	3	2	4	3	3	4	3	3	4	4	$(b_1 \oplus b_4, 0)$	(b_1, b_2)	(b_3, b_4)	(b_1, b_4)
4	2	2	2	2	4	4	3	4	3	3	5	4	4	4	5	(b_1, b_2)	(b_3, b_4)	$(b_5, b_1 \oplus b_3)$	(b_1, b_3)
1	2	2	2	2	4	4	4	4	4	4	4	4	4	4	4	(b_1, b_2)	(b_3, b_4)	$(b_1 \oplus b_3, b_2 \oplus b_4)$	$(b_2 \oplus b_3, b_1 \oplus b_2 \oplus b_4)$
12	2	2	1	2	3	2	3	2	3	3	3	3	3	3	3	(b_1, b_2)	$(b_3, b_1 \oplus b_2)$	$(b_1 \oplus b_2, 0)$	$(b_3 \oplus b_1, b_3 \oplus b_2)$
12	2	2	2	2	3	3	3	3	3	4	4	3	4	4	4	(b_1, b_4)	(b_1, b_3)	(b_1, b_2)	(b_3, b_4)
4	1	1	1	2	2	2	3	2	3	3	3	3	3	3	3	$(b_1, 0)$	$(b_2, 0)$	$(b_1 \oplus b_3, 0)$	$(b_1 \oplus b_2, b_3)$
4	2	2	2	2	3	3	4	3	4	4	4	4	4	4	4	$(b_1, b_2 \oplus b_3)$	$(b_1, b_2 \oplus b_4)$	(b_1, b_2)	(b_3, b_4)
12	1	1	2	2	2	3	3	3	3	4	4	3	4	4	4	$(b_1 \oplus b_3, 0)$	$(b_1 \oplus b_3 \oplus b_4, 0)$	(b_1, b_2)	(b_3, b_4)
12	1	2	2	2	3	3	3	4	4	4	5	4	4	5	5	$(b_2 \oplus b_3, 0)$	$(b_5, b_2 \oplus b_4)$	(b_1, b_2)	(b_3, b_4)
4	2	2	2	2	4	4	4	4	4	4	6	5	5	5	6	(b_1, b_2)	(b_3, b_4)	(b_5, b_6)	$(b_2 \oplus b_6, b_4 \oplus b_6)$
4	2	2	2	2	3	3	3	4	4	4	4	4	4	4	4	$(b_1 \oplus b_2, b_3 \oplus b_4)$	$(b_1 \oplus b_3, b_2 \oplus b_4)$	(b_1, b_2)	(b_3, b_4)
12	2	2	2	2	3	3	3	3	4	4	4	4	4	4	4	(b_1, b_3)	$(b_1, b_2 \oplus b_4)$	(b_1, b_2)	(b_3, b_4)
12	1	2	2	2	3	3	3	3	3	4	4	4	4	4	4	$(b_1 \oplus b_3, 0)$	(b_2, b_4)	(b_1, b_2)	(b_3, b_4)
12	2	2	2	2	4	4	4	3	3	4	5	5	5	4	5	(b_1, b_2)	(b_3, b_4)	(b_5, b_3)	$(b_1 \oplus b_5, b_4)$

TABLE II

THE FORMS OF SHANNON OUTER BOUND EXTREME POINTS THAT (WHEN SCALED BY 2) ARE ACHIEVABLE USING TWO BITS PER RANDOM VARIABLE, AND EXAMPLE CONSTRUCTIONS FOR THEM USING UNIFORM BITS $b_1, b_2, b_3, b_4, b_5, b_6$.

#	h_1	h_2	h_3	h_4	h_{12}	h_{13}	h_{14}	h_{23}	h_{24}	h_{34}	h_{123}	h_{124}	h_{134}	h_{234}	h_{1234}
6	2	2	2	2	3	3	3	3	3	4	4	4	4	4	4

TABLE III

THE FORM OF SHANNON OUTER BOUND EXTREME POINTS THAT VIOLATES THE INGLETON INEQUALITY AND IS NOT ENTROPIC.

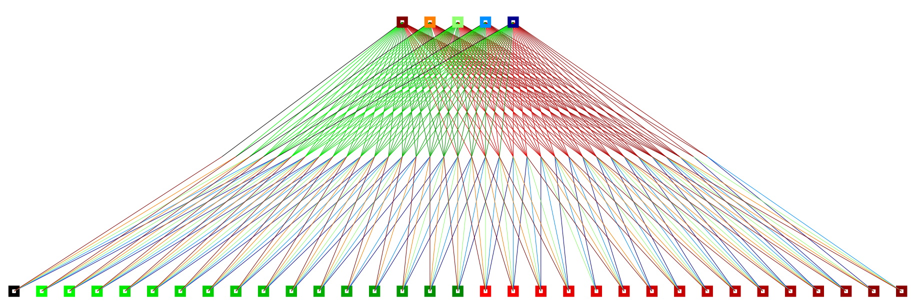

Fig. 3. Interdependence of extreme points and exposed faces of the Shannon outer bound $\Gamma_4 \cap \mathcal{B}_4$. The bottom nodes represent the (equivalence classes under variable label permutations of) extreme points of the Shannon outer bound, while the top nodes represent its (equivalence classes under variable label permutations of) exposed faces. The 16 extreme points which are different shades of red represent entropy vectors achievable using only binary random variables, while the 16 extreme points which are colored in different shades of green represent entropy vectors achievable using two bits for each random variable. The 1 black extreme point is not entropic.

order permutations of 33 forms of extreme points. These 33 extreme point forms consist of 16 extreme point forms (67 extreme points) which are achievable with binary random variables listed in Table I, 16 extreme point forms (133 extreme points) which are achievable (when scaled by 2) with two bit random variables listed in Table II, and 1 extreme point form (6 extreme points) which is unachievable and violates the Ingleton inequality listed in Table III.

The constructions given in Tables I and II use only *uniform* bits, i.e., binary random variables that are equally likely to take their two values. This is not too surprising in light of prior observations in the literature that all of the extreme points of the entropy region must be associated with quasi-uniform distributions, e.g., [3], [4], [5], [13], [14].

It is interesting to study which (forms of) extreme points are incident on which (forms of) exposed faces. Fig. 3 presents this information in the form of a colored bipartite graph. The bottom nodes represent the 33 forms of the 206 extreme extreme points of the Shannon outer bound. The top nodes represent the 5 forms of the 32 (non-redundant) exposed faces of the Shannon outer bound. The exposed faces and extreme points are ordered and colored according to their form: two different extreme points that are different permutations of the same form are marked with the same color. The black extreme points are the Ingleton violators, while the green extreme points are the two bit achievable extreme points, and the red extreme points are the one bit achievable extreme points.

One can draw from this figure several important conclusions:

- Every facet of the Shannon outer bound has at least one extreme point that is Ingleton violating, even though there are only 6 Ingleton violating Shannon extreme points.
- Every facet of the Shannon outer bound has some extreme points which can be achieved with just binary random variables, while at the same time having some extreme points which can be achieved with two bit random variables (but not one bit). The binary achievable extreme points' convex hull yields an inner bound which has a (full 14-dimensional) facet that is a subset of every Shannon outer bound facet. A larger inner bound also having this property is the convex hull of the set of one and two bit achievable Shannon extreme points.
- *All* of the entropic extreme points of *every* facet of the Shannon outer bound for $\bar{\Gamma}_4^* \cap \mathcal{B}_4$ can be achieved using only two bit random variables.
- As the extreme points of every Shannon facet contain at least one Ingleton violating extreme point, we see that there is room for improvement on *every* facet of the Shannon outer bound.

These observations may also be stated as follows: $i)$ most (200 of 206) Shannon outer bound extreme points satisfy the Ingleton inequality, while $ii)$ all (32 of 32) Shannon outer bound facets contain at least one extreme point that violates the Ingleton inequality. In other words, the lowest dimension faces ($d = 0$, extreme points) are mostly free of Ingleton violation, while the highest dimension faces ($d = 14$, facets) all contain Ingleton violating points. This raises the question about the pervasiveness of Ingleton violation at intermediate dimension faces. To investigate this, we note that any polytope face at any dimension is the convex hull of a subset of the 206 Shannon extreme points. Since all Shannon extreme points are of three types (binary achievable, two bit achievable, or Ingleton violating), we can classify each subset of extreme points as one of seven types:

type	meaning
1	all extreme points binary achievable
2	all extreme points two-bit achievable
1,2	all extreme points one or two bit achievable
I	all extreme points Ingleton violators
1,I	all extreme points binary ach. or Ingleton violator
2,I	all extreme points two-bit ach. or Ingleton violator
1,2,I	all three types of extreme points present

For example, if a subset is of type $1, I$ then all extreme points are either binary achievable or Ingleton violators, and there are no two-bit achievable extreme points in the set. With this notation in hand, for each dimension we count the number of faces of each type and the fraction of faces of each dimension of each type in Table IV (next page). Several observations merit comment:

- As is typical in a Hasse diagram, the number of intermediate dimensional faces of the Shannon outer bound is quite large in comparison to the number of facets and extreme points.
- A significant fraction of intermediate dimensional faces are have extreme points which are only binary achievable or Ingleton violating.
- For all but the highest dimension (i.e. all but the facets) a significant fraction of the Shannon faces are correct, and can be achieved with either one or two bits per variable. This suggests, together with results presented later in the paper, that the investigation into entropy vectors associated with low cardinality discrete random variables should provide good estimates of the region of entropic vectors.

To summarize the original question regarding the pervasiveness of the Ingleton violators at intermediate dimensions, we label a face as "good" if its extreme points do not contain an Ingleton violator, and "bad" if its extreme points contain one or more Ingleton violators. Table V gives the number and fraction of good and bad faces at each dimension. Note the fraction of good faces decreases steadily in the dimension.

B. The Ingleton Inner Bound

The Ingleton inner bound keeps the 200 binary and two bit achievable Shannon extreme points (which are also extreme points of $\bar{\Gamma}_4^* \cap \mathcal{B}_4$) of 32 forms, and replaces the 6 Ingleton violating extreme points with the 6 extreme points that are permutations of the extreme point form (scaled by 3) shown in Table VI. These extreme points are achievable using, e.g.,

D	N	N_1	N_2	N_{12}	N_I	N_{1I}	N_{2I}	N_{12I}	f_1	f_2	f_{12}	f_I	f_{1I}	f_{2I}	f_{12I}
0	206	67	133	0	6	0	0	0	0.325	0.646	0	0.029	0	0	0
1	3998	1396	416	1844	0	270	72	0	0.349	0.104	0.461	0	0.068	0.018	0
2	32182	11755	408	16083	0	2856	144	936	0.365	0.013	0.500	0	0.089	0.004	0.029
3	137372	47386	152	68876	0	14046	96	6816	0.345	0.001	0.501	0	0.102	0.001	0.050
4	358089	107703	16	184887	0	39831	24	25628	0.301	0	0.516	0	0.111	0	0.072
5	621389	151500	0	335533	0	72114	0	62242	0.244	0	0.540	0	0.116	0	0.100
6	757224	139364	0	425024	0	87984	0	104852	0.184	0	0.561	0	0.116	0	0.138
7	669467	85698	0	382818	0	74262	0	126689	0.128	0	0.572	0	0.111	0	0.189
8	437253	34950	0	247103	0	43584	0	111616	0.080	0	0.565	0	0.100	0	0.255
9	212364	9050	0	113780	0	17496	0	72038	0.043	0	0.536	0	0.082	0	0.339
10	76360	1336	0	36596	0	4584	0	33844	0.017	0	0.479	0	0.060	0	0.443
11	19986	84	0	7843	0	705	0	11354	0.004	0	0.392	0	0.035	0	0.568
12	3680	0	0	1012	0	48	0	2620	0	0	0.275	0	0.013	0	0.712
13	448	0	0	60	0	0	0	388	0	0	0.134	0	0	0	0.866
14	32	0	0	0	0	0	0	32	0	0	0	0	0	0	1.000

TABLE IV

NUMBER AND FRACTION OF FACES PER DIMENSION OF COMPRISED OF BINARY ACHIEVABLE (1), TWO-BIT ACHIEVABLE (2), AND INGLETON VIOLATING (I) EXTREME POINTS.

#	$h_1h_2h_3h_4$	$h_{12}h_{13}h_{14}h_{23}h_{24}h_{34}$	$h_{123}h_{124}h_{134}h_{234}$	h_{1234}	X_1	X_2	X_3	X_4
					$b_2 \oplus b_3 \oplus b_5$	b_1	$b_1 \oplus b_4 \oplus b_5 \oplus b_6$	b_4
6	3 3 3 3	5 4 5 5 6 5	6 6 6 6	6	$b_4 \oplus b_5$	b_2	$b_4 \oplus b_5$	b_5
					$b_3 \oplus b_5$	b_3	b_2	b_6

TABLE VI

THE ONLY EXTREME POINT FORM WHICH THE INGLETON INNER BOUND ADDS TO THE INGLETON OBEYING SHANNON EXTREME POINTS TO GET ITS EXTREME POINTS AND ITS THREE BIT PER RANDOM VARIABLE CONSTRUCTION.

h_1	h_2	h_3	h_4	h_{12}	h_{13}	h_{14}	h_{23}	h_{24}	h_{34}	h_{123}	h_{124}	h_{134}	h_{234}	h_{1234}
0.9863	0.9992	0.9529	0.9999	1.8855	1.9373	1.9802	1.9521	1.9840	1.6962	2.7985	2.8690	2.6763	2.6711	3.4697

TABLE VII

A BINARY ENTROPY VECTOR WHICH VIOLATES THE INGLETON INEQUALITY.

	Binary Ach. Shan.	All Entropic Shan.	Ingleton Inner Bound
Shan. Outer	0.3982	$\frac{1}{2\sqrt{10}} \approx .1581$	$\frac{1}{2\sqrt{10}} \approx .1581$
DFZ & YZ + Shan.	0.3982	$\frac{1}{3\sqrt{10}} \approx .1054$	$\frac{1}{3\sqrt{10}} \approx .1054$

TABLE VIII

WORST CASE GAP BETWEEN INNER AND OUTER BOUNDS FOR $\bar{\Gamma}_4^* \cap \mathcal{B}_4$. THE FIRST COLUMN IS THE CONVEX HULL OF THOSE ENTROPIC SHANNON EXTREME POINTS WHICH CAN BE ACHIEVED WITH ONE BIT, THE SECOND COLUMN IS THE CONVEX HULL OF ALL OF THE ENTROPIC SHANNON EXTREME POINTS (THESE ALL CAN BE ACHIEVED WITH ONE OR TWO BITS), WHILE THE THIRD COLUMN IS THE INGLETON INNER BOUND.

dim	num	num good	num bad	% good	% bad
0	206	200	6	0.971	0.029
1	3998	3656	342	0.914	0.086
2	32182	28246	3936	0.878	0.122
3	137372	116414	20958	0.847	0.153
4	358089	292606	65483	0.817	0.183
5	621389	487033	134356	0.784	0.216
6	757224	564388	192836	0.745	0.255
7	669467	468516	200951	0.700	0.300
8	437253	282053	155200	0.645	0.355
9	212364	122830	89534	0.578	0.422
10	76360	37932	38428	0.497	0.503
11	19986	7927	12059	0.397	0.603
12	3680	1012	2668	0.275	0.725
13	448	60	388	0.134	0.866
14	32	0	32	0	1.000

TABLE V

NUMBER AND PERCENTAGE OF FACES PER DIMENSION WITH (BAD) AND WITHOUT (GOOD) INCIDENT INGLETON VIOLATING EXTREME POINTS.

the three bit per random variable construction listed to the right of the entropy extreme point form.

From these facts we can discern the following important conclusions:

- The Ingleton inner bound is the *best* inner bound that can be computed using a convex hull of Shannon outer bound $\bar{\Gamma}_4^*$ extreme points. That is, the Ingleton inner bound contains all of the entropic Shannon outer bound extreme points.
- The Ingleton inner bound for entropy vectors in four variables is *contained within* the convex hull of entropy vectors achievable with 3 bits per variable.

Following up on the last idea, it should be noted that it is quite easy to produce distributions on four bits $X_1, \ldots, X_4$ which violate the Ingleton inequality. One such distribution (chosen at random) gives the binary entropy vector listed in Table VII. The existence of a binary entropic Ingleton violator shows that in fact that the containment between the convex hull of 3 bit per variable entropic vectors and the

Ingleton inner bound is strict.

Hence, we observe that while $\mathrm{conv}(\Phi_4)$ and the Ingleton inner bound intersect (their intersection contains the convex hull of the 67 binary achievable Shannon extreme points), there is no containment relation in either direction between these two inner bounds for $\bar{\Gamma}_4^*$. These relationships are pictured in Fig. 1 and Fig. 4.

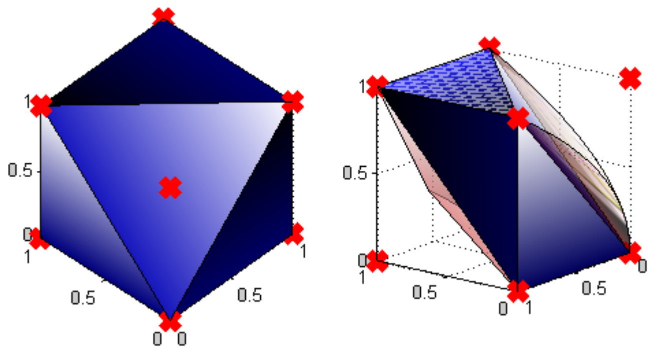

Fig. 4. Pictoral representation of some of the results in this paper. **Left:** the convex hull of six of the eight extreme points of the cube, where the two absent extreme points are maximally separated. The resulting convex body has the property that each of the six facets of the cube has a full-dimensional intersection with a facet from the convex body, but none of the six facets are shared, i.e., each of the six facets of the cube contains one of the absent extreme points. This is analogous to the relationship between the Shannon outer bound and the Ingleton inner bound: each facet of the Shannon outer bound has a full-dimensional intersection with a facet from the Ingleton inner bound, but each such facet also contains the Ingleton violating Shannon extreme point, which is of course not present in the inner bound. **Right:** a rotated view of the same figure as left, with some "curvature" added.

C. The Zhang & Yeung + Dougherty, Freiling, & Zeger Outer Bound

The outer bound to $\bar{\Gamma}_4^* \cap \mathcal{B}_4$ formed by all Shannon-type inequalities in addition to the non-Shannon type inequalities of Zhang & Yeung [2] and Dougherty, Freiling, & Zeger [10] (denoted as the DFZ+ZY bound) consists of 2744 extreme points which are variable label permutations of 152 forms. These consist of the 200 binary and two bit achievable Shannon extreme points (which each consist of 16 different forms), in addition to 120 new extreme point forms. While these are too many too list here, it is interesting to point out a couple of relevant properties of these new outer bound extreme points.

- Each of these 120 new extreme point forms violates an Ingleton inequality.
- None of these 120 new extreme point forms are binary achievable.

That is, no new extreme points of the set of binary entropy vectors are found with this outer bound.

IV. Worst case gap between currently available bounds for the entropy region

Given a polytopic inner bound $\mathcal{I}$ and polytopic outer bound $\mathcal{O}$ to a set $\mathcal{G}$, we can find the worst case gap between these bounds as the square root of

$$\max_{\mathbf{x}\in\mathcal{O}} \min_{\mathbf{y}\in\mathcal{I}} \|\mathbf{x}-\mathbf{y}\|_2^2$$

Because $\|\mathbf{x}-\mathbf{y}\|_2^2$ is convex, and because convexity is preserved under minimization, the function

$$f(\mathbf{x}) = \min_{\mathbf{y}\in\mathcal{I}} \|\mathbf{x}-\mathbf{y}\|_2^2$$

is convex and can easily be found numerically. Furthermore, because every point $\mathbf{x}\in\mathcal{O}$ can be represented as a convex combination of the set of extreme points $\{\mathbf{v}_i\}$ of $\mathcal{O}$,

$$\begin{aligned}
\max_{\mathbf{x}\in\mathcal{O}} f(\mathbf{x}) &= \max_{\boldsymbol{\lambda}\geq 0,\ \mathbf{1}^T\boldsymbol{\lambda}=1} f\left(\sum_i \lambda_i \mathbf{v}_i\right) && (2)\\
&\leq \max_{\boldsymbol{\lambda}\geq 0,\ \mathbf{1}^T\boldsymbol{\lambda}=1} \sum_i \lambda_i f(\mathbf{v}_i) && (3)\\
&= \max_i f(\mathbf{v}_i) && (4)
\end{aligned}$$

which is attained at $\mathbf{x}=\mathbf{v}_i$. Thus, we can compute the worst case gap between the two bounds as the square root of

$$\max_{\mathbf{v}\text{ an extreme point of }\mathcal{O}} \ \min_{\mathbf{y}\in\mathcal{I}} \|\mathbf{v}-\mathbf{y}\|_2^2$$

Given the dual representations of the available bounds for $\bar{\Gamma}_4^* \cap \mathcal{B}_4$, we can use this trick to easily calculate the worst case gap between pairs of inner and outer bounds. The results are presented in Table VIII, and are somewhat surprising. In particular, although the Ingleton inner bound strictly contains the convex hull of the entropic Shannon extreme points, its addition of an extra entropic extreme point does not improve the worst case gap to the Shannon or DFZ outer bounds.

V. Conclusion

By considering the extreme point representation of the Shannon outer bound for the region of entropy vectors for 4 random variables $\bar{\Gamma}_4^* \cap \mathcal{B}_4$, we were able to find 200 extreme points of $\bar{\Gamma}_4^*$ of 32 forms. We saw that all entropic Shannon extreme points are achievable with either one or two bits per random variable. We saw that the Shannon outer bound has only 6 non-entropic extreme points, and these were all permutations of one form which violates the Ingleton inequality. Inspecting which extreme points neighbored which exposed faces enabled us to determine that *every* Shannon facet can be improved. The fraction of Shannon faces which can be improved upon was shown to be slowly decreasing in the face dimension. We saw that the Ingleton inner bound shares all of the entropic Shannon extreme points, and adds six more extreme points which are all permutations of one form which is achievable using three bits per random variable. We saw that there is containment in neither direction between the Ingleton inner bound, and the convex hull of the set of binary entropy vectors. Together, these imply that the Ingleton inner bound is strictly contained within the convex hull of the set of three bit entropic vectors. We saw that while the outer bound formed by including Zhang and Yeung and Dougherty, Freiling and Zeger's non-Shannon information inequalities adds 120 new extreme point forms that are all Ingleton violating to the 32 entropic Shannon extreme point forms, all of these new extreme points are not binary achievable. Finally, we noted that the worst case distance gap between the Ingleton inner bound and the Shannon or DFZ outer bounds is no better than the worst case distance gap achieved with the inner bound formed as the convex hull of the entropic Shannon extreme points.

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