ECES 811 Homework 3 Solutions

Yunshu Liu

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1. For each set below, determine whether or not it is convex and prove your answer
   (a) $\left\{(x,y)\,\middle|\,y \geq x^4,\ 2 \geq x \geq -2\right\}$
   This set is convex. It is the intersection of two convex sets.
   (b) $\{(x,y)\,|y \geq \exp(x-1)\}$
   This set is convex. It is the epigraph of a convex function $\exp(x-1)$ which is easily seen as strictly convex by taking the second derivative and observing that it is > 0 for all $x \in \mathbb{R}$.
   (c) $\{(x,y)\,|y \leq x(x-2)(x-3)\}$
   This set is not convex. Consider the two points $(2,0)$ and $(3,0)$, which are in this set, however, the convex combination $(2\lambda + 3(1-\lambda), 0)$ is not in the set for some $0 < \lambda < 1$.
   (d) $\left\{(x,y)\,\middle|\,x^2 + y^2 \leq 4,\ x \geq 0\right\}$
   This set is convex. It is the intersection of two convex set.

2. Find the extreme points and extreme directions of each of the convex sets from problem # 1.
   (a) The extreme points: $\{(x,y)|y = x^4, -2 \leq x \leq 2\}$ for $x \in \mathbb{R}$.
   The extreme directions: $\{(-2,y)|y \geq 0\}$ and $\{(2,y)|y \geq 0\}$.
   (b) The extreme points: $\{(x,y)|y = \exp(x-1)\}$ for $x \in \mathbb{R}$. This set has no faces which are halflines, and thus has no extreme directions.
   (c) Not convex
   (d) The extreme points :$(x,y) = (2\sin(\theta), 2\cos(\theta))$ for $\theta \in [0,\pi]$. This set has no extreme directions.

3. Prove that the minimum of a linear function $\mathbf{c}^T\mathbf{x}$ over $\mathbf{x} \in \mathcal{C}$ for a compact (closed and bounded) convex set $\mathcal{C}$ must be attained at an extreme point of $\mathcal{C}$.
   Proof: Since $\mathcal{C}$ is a compact (closed and bounded) convex set, it must have extreme point. Assuming the minimum of $\mathbf{c}^T\mathbf{x}$ is obtained at a point x_0 which is not an extreme point of $\mathcal{C}$, there exist $\epsilon > 0$ such that both $x_0 - \epsilon \in \mathcal{C}$ and $x_0 + \epsilon \in \mathcal{C}$. Because of the assumption, we have $\mathbf{c}^T x_0 \leq \mathbf{c}^T(x_0 - \epsilon)$ and $\mathbf{c}^T x_0 \leq \mathbf{c}^T(x_0 + \epsilon)$, which give us $\mathbf{c}^T\epsilon \leq 0$ and $\mathbf{c}^T\epsilon \geq 0$, contradict with $\epsilon > 0$ (Note:Another approach to this problem is similar to the proof of the problem 4)

4. Prove that the maximum of a convex function $f(\mathbf{x})$ over $\mathbf{x} \in \mathcal{C}$ for a compact (closed and bounded) convex set $\mathcal{C}$ must be attained at an extreme point of $\mathcal{C}$.
   Proof: Assuming the maximum of $f(\mathbf{x})$ is attained at a point y_0 which is not an extreme point of $\mathcal{C}$. For the compact convex set $\mathcal{C}$, there must exist a set of extreme points $x_1, x_2, \cdots, x_m$ of $\mathcal{C}$ and a set of parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$ satisfing $0 < \lambda_1, \lambda_2, \cdots, \lambda_m < 1$ and $\sum_{i=1}^m \lambda_i = 1$ such that $y_0 = \sum_{i=1}^m \lambda_i x_i$. From the definition of convex function, we have $f(y_0) = f(\sum_{i=1}^m \lambda_i x_i) \leq \sum_{i=1}^m \lambda_i f(x_i)$, which infers $f(y_0) \leq \max_{i=1,\cdots,m} f(x_i)$ with equality if and only if $f(y_0) = f(x_1) = \cdots = f(x_m)$. This contradict with our previous assumption.