Information Projection Algorithms and Belief Propagation

Phil Regalia

Department of Electrical Engineering and Computer Science
Catholic University of America
Washington, DC 20064

with J. M. Walsh, Drexel University

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Outline

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2. Bregman Distance
3. Iterative Projection Algorithms
4. Belief Propagation
5. Conclusion
Belief Propagation $\iff$ Iterative Convex Projection algorithms

Belief Propagation (Pearl, 1986) has met with success in:

- Error correction decoding (low density parity-check codes, turbo codes, ...);
- Network diagnostics and link monitoring;
- Sensor self-localization;
- Distributed estimation in sensor networks;
- Lossy source quantization;
- Multi-user communications;
- Et cetera.

Iterative Convex Projection algorithms have convergence proofs, unlike BP for which proofs assume either:

- Tree or forest dependency graph;
- Arbitrarily large factor graph girth.
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Basic Query

Can iterative convex projection algorithms lend insight into belief propagation?

A priori yes, and the role of information projections and information geometry in BP is nothing new:

- Moher and Gulliver (Trans. IT-98) rephrased iterative decoding as information projections of Csiszár;
- Grant (ISIT-99) examined turbo decoding via information projections;
- Richardson (Trans. IT-03) viewed nonlinear dynamics of iterative decoding via (tacitly) information geometry;
- Ikeda, Tanaka and Amari (Trans. IT-04) rephrased “all of the above” in terms of information geometry.

Key obstacle: “Extrinsic information extraction” goes against projections on invariant sets.
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Bregman Distance

Graph of a convex function $f(\cdot)$ is lower bounded by any tangent hyperplane:

$$\begin{align*}
f(r) & \geq f(q) + \langle \nabla f(q), r - q \rangle \\
& \geq f(q) + \langle \nabla f(q), r - q \rangle
\end{align*}$$

This induces the gradient inequality:

$$f(r) \geq f(q) + \langle \nabla f(q), r - q \rangle$$

whose discrepancy in turn induces the Bregman distance

$$D_f(r, q) = f(r) - f(q) - \langle \nabla f(q), r - q \rangle \geq 0$$
Common examples

Probability mass functions defined on $M$ outcomes:

$$q_i = \Pr(x = x_i), \quad i = 0, 1, 2, \ldots, M-1.$$ 

Introduce convex domain

$$\mathcal{D} = \left\{ q_i : \begin{array}{l} q_1 \geq 0 \\ \vdots \\ q_{M-1} \geq 0 \\ q_1 + q_2 + \cdots + q_{M-1} \leq 1 \end{array} \right\}$$

Setting $q_0 = 1 - \sum_{i=1}^{M-1} q_i$, the negative Shannon entropy

$$f(q) = \left( 1 - \sum_{i=1}^{M-1} q_i \right) \log \left( 1 - \sum_{i=1}^{M-1} q_i \right) + \sum_{i=1}^{M-1} q_i \log q_i$$

is convex over $\mathcal{D}$. 
The induced Bregman distance becomes

\[ D_f(r, q) = \sum_{i=0}^{M-1} r_i \log \frac{r_i}{q_i} \]

and is recognized as the Kullback-Leibler divergence.

Closely related is the Fenchel conjugate function

\[ f^*(\theta) \triangleq \sup_q \left( \langle q, \theta \rangle - f(q) \right) \]

\[ = \log \left( \sum_{i=0}^{M-1} \exp(\theta_i) \right), \quad \text{with} \quad \theta_i = \log \frac{q_i}{q_0} \]

This is convex over a domain \( \mathcal{D}^* \subset \mathbb{R}^M \) of vectors having first component zero. ("Log probability ratios")
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This is convex over a domain \( D^* \subset \mathbb{R}^M \) of vectors having first component zero. ("Log probability ratios")
Induced Bregman distance is now

\[
D_{f^*}(\rho, \theta) = \log \frac{\sum_{i=0}^{M-1} \exp(\rho_i)}{\sum_{i=0}^{M-1} \exp(\theta_i)} - \sum_{i=0}^{M-1} \frac{\exp(\theta_i)(\rho_i - \theta_i)}{\sum_{j=0}^{M-1} \exp(\theta_j)}
\]

By associating

\[
q_i = \frac{\exp(\theta_i)}{\sum_{j=0}^{M-1} \exp(\theta_j)} \quad \quad r_i = \frac{\exp(\rho_i)}{\sum_{j=0}^{M-1} \exp(\rho_j)}
\]

This assumes the more familiar form

\[
D_{f^*}(\rho, \theta) = \sum_{i=0}^{M-1} q_i \log \frac{q_i}{r_i}
\]
Observe that
\[
\left[ \nabla f(q) \right]_i = \frac{df(q)}{dq_i} = \log \frac{q_i}{q_0} = \theta_i
\]
\[
\left[ \nabla f^*(\theta) \right]_i = \frac{df^*(\theta)}{d\theta_i} = \frac{\exp(\theta_i)}{\sum_i \exp(\theta_i)} = q_i
\]
giving inverse maps, as \( f(q) \) and \( f^*(\theta) \) are Legendre transforms of each other.

In general, for convex \( f \) and its conjugate \( f^* \) in the Legendre class, their induced Bregman distances satisfy

\[
D_f(r, q) = D_{f^*}\left( \nabla f(q), \nabla f(r) \right).
\]
Consider finally the energy function:

\[ f(q) = \frac{1}{2} \sum_{i=0}^{M-1} q_i^2, \quad q \in \mathbb{R}^M \]

the Bregman distance becomes

\[ D_f(r, q) = \frac{1}{2} \sum_{i=0}^{M-1} (r_i - q_i)^2. \]

As \( \nabla f(q) = q \), and \( f^* = f \), this gives a symmetric Bregman distance:

\[ D_f(r, q) = D_f(q, r) \]
If $f(q)$ is a convex function over a domain $\mathcal{D}$, and $\mathcal{C}$ is a convex subset of $\mathcal{D}$, the Bregman projection of $q$ onto $\mathcal{C}$ is

$$\pi_{\mathcal{C}}(q) = \arg \min_{r \in \mathcal{C}} D_f(r, q).$$

and is characterized by the inequality

$$D_f(r, q) \geq D_f\left(r, \pi_{\mathcal{C}}(q)\right) + D_f\left(\pi_{\mathcal{C}}(q), q\right), \quad \text{for all } r \in \mathcal{C},$$

or, equivalently,

$$\langle \nabla f(q) - \nabla f(\pi_{\mathcal{C}}(q)), r - \pi_{\mathcal{C}}(q) \rangle \leq 0, \quad \text{for all } r \in \mathcal{C}.$$
Dykstra’s Cyclic Projection Algorithm

Seek minimization

$$\pi_C(q) = \arg\min_{r \in C} D_f(r, q)$$

where $C$ is the intersection of convex sets: $C = \bigcap_{n=1}^N C_n$

First stab: Sometimes convergent algorithm,

$$r_n = \pi_{C_n}(r_{n-1})$$

$$= \pi_{C_n}(\nabla f^*(\nabla f(r_{n-1})))$$

using $C_{n+N} = C_n$ and initialization

$$r_0 = q,$$
Dykstra’s Cyclic Projection Algorithm

Seek minimization

$$\pi_C(q) = \arg \min_{r \in C} D_f(r, q)$$

where $C$ is the intersection of convex sets: $C = \bigcap_{n=1}^{N} C_n$

Improved algorithm, $r_n \xrightarrow{n \to \infty} \pi_C(q)$:

$$r_n = \pi_{C_n}\left(\nabla f^*(\nabla f(r_{n-1}) + s_{n-N})\right)$$

$$s_n = \nabla f(r_{n-1}) + s_{n-N} - \nabla f(r_n)$$

using $C_{n+N} = C_n$ and initialization

$$r_0 = q, \quad s_{-(N-1)} = \cdots = s_{-1} = s_0 = 0.$$
Minimum Distance Algorithm

Find closest members:

\[ D_f(r_*, q_*) = \inf_{r \in C_1, q \in C_2} D_f(r, q), \quad \text{where } C_1 \cap C_2 = \emptyset. \]

Cyclic projection algorithm becomes

\[
\begin{align*}
    r_n &= \pi_{C_1} \left( \nabla f^* \left( \nabla f(q_{n-1}) + v_{n-1} \right) \right) \\
    v_n &= \nabla f(q_{n-1}) + v_{n-1} - \nabla f(r_n) \\
    q_n &= \pi_{C_2} \left( \nabla f \left( \nabla f^*(r_n) + w_{n-1} \right) \right) \\
    w_n &= \nabla f^*(r_n) + w_{n-1} - \nabla f^*(q_n)
\end{align*}
\]

Convergent in the Euclidean case

\[ D_f(r, q) = \frac{1}{2} \sum_k (r_k - q_k)^2. \]
Belief Propagation

Iterative (sometimes “fast”) algorithm to calculate marginal probability functions.

Given binary vector \( \mathbf{x} = [x_1, \ldots, x_M] \in \{0, 1\}^M \), and probability function \( G(\mathbf{x}) \), seek marginals

\[
f_k(x_k) = \sum_{x_1=0}^1 \cdots \sum_{x_{k-1}=0}^1 \sum_{x_{k+1}=0}^1 \cdots \sum_{x_M=0}^1 G(\mathbf{x})
\]

Calculation is “hard” since there are \( 2^M \) evaluations for \( \mathbf{x} \).

BP is applicable when \( G(\mathbf{x}) \) splits into simpler factors:

\[
G(\mathbf{x}) = \prod_{k=1}^{K} g_k(\mathbf{x})
\]

Usually, each factor \( g_k \) depends only on a subset of variables in \( \mathbf{x} \).
Message passing algorithm

\[ m_{x_i \rightarrow g_k}(x_i) = \text{variable } x_i \text{ to factor } g_k \]
\[ = \beta_k \prod_{\ell \neq k} m_{g_\ell \rightarrow x_i}(x_i) \]

\[ m_{g_k \rightarrow x_i}(x_i) = \text{factor } g_k \text{ to variable } x_i \]
\[ = \alpha_i \sum_{x_n: n \neq i} g_k(x) \prod_{\ell \neq i} m_{x_\ell \rightarrow g_k}(x_\ell) \]

Outgoing message on an edge depends only on incoming messages from other edges at that node ("extrinsic" information).

If convergent, the beliefs

\[ b_i(x_i) = \beta_i \prod_{\ell} m_{g_\ell \rightarrow x_i}(x_i) \]

are thresholded: \( b_i(0) \geq b_i(1) \). Ideally, beliefs would be marginals.
For projection interpretation, let $x_1, \ldots, x_K$ be $K$ copies of $x$, and consider extended likelihood function

$$G(x_1, \ldots, x_K) = \prod_{k=1}^{K} g_k(x_k).$$

Original function is obtained with constraint $x_1 = \cdots = x_K = x$.

Two convex sets:

- Product distributions of $2^{MK}$ variables:

$$\mathcal{P} = \left\{ \theta : \nabla f^*(\theta) = \prod_{k=1}^{K} \prod_{m=1}^{M} q_{k,m}(x^k_m) \right\}$$

- Constraint (or "sparse") distributions

$$Q = \left\{ r : r(x^1, \ldots, x^K) = 0 \text{ if } x^i \neq x^j \text{ for any } i \neq j \right\}$$
Information geometric view:

- **At factor nodes**: Given pmf \( q \), if \( r \in \mathcal{P} \) is a product distribution built from the marginals of \( q \), then

\[
D_f(q, s) = D_f(q, r) + D_f(r, s), \quad \text{for all } s \in \mathcal{P}
\]

so that \( r \) is the Bregman projection of \( q \) onto \( \mathcal{P} \).

- **At variable nodes**: Given pmf \( q \), if \( Q \) denotes a set of “sparse” distributions, then

\[
\beta_i q_i, \quad i \in \text{index set for } Q; \\
0, \quad \text{otherwise;}
\]

satisfies

\[
D_f(s, q) = D_f(s, b) + D_f(b, q), \quad \text{for all } s \in Q.
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so that \( b \) (containing beliefs) is Bregman projection onto \( Q \).
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so that \( b \) (containing beliefs) is Bregman projection onto \( Q \).
Calculation of marginal pmfs

"Ideal" solution is

\[ p = \pi_Q(\nabla f^*(\chi_{-1})) \quad \text{(constrain to sparse distributions)} \]

\[ \chi_0 = \pi_P(\nabla f(p)) \quad \text{(calculate marginal distributions)} \]

with "initialization"

\[ \chi_{-1} = \nabla f\left(\prod_k g_k(x_k)\right) \]
Belief propagation is the cyclic projection algorithm

\[
\begin{align*}
\xi_n &= \pi_P(\chi_{n-1} + \sigma_{n-1}) \\
\sigma_n &= \chi_{n-1} + \sigma_{n-1} - \xi_n \\
p_n &= \pi_Q(\nabla f^*(\xi_n + \tau_{n-1})) \\
\chi_n &= \pi_P(\nabla f(p_n)) \\
\tau_n &= \xi_n + \tau_{n-1} - \chi_n
\end{align*}
\]

with initialization

\[
\begin{align*}
\chi_{-1} &= \nabla f\left(\prod_k g_k(x_k)\right) \\
\sigma_{-1} &= \tau_{-1} = 0
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\]
**Special Case:** Replace negative Shannon entropy by energy function, to get “Euclidean belief propagation”:

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\end{align*}
\]

Given arbitrary convex sets $P$ and $Q$, can show that $\{\chi_n\}$ converges for any initial condition $\chi_{-1}$.

Conventional belief propagation, by contrast, is convergent when either of the following apply:

- Factor graph is a tree/forest (or nearly so: large girth);
- Initialization $\chi_{-1}$ is a product distribution (or nearly so: $\pi_P(\chi_{-1}) \approx \chi_{-1}$).
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Further perspectives:

- Rephrasing belief propagation in terms of cyclic convex projection algorithms suggests that convergence studies of the latter might extend to the former.

- “Euclidean” belief propagation is always convergent, although conventional (information projection-based) belief propagation can diverge in some settings.

- Continuity of projectors can be invoked to extend cases where belief propagation is proved to converge to “good” solutions.

- Can some modification to belief propagation give a more faithful transcription of Dykstra’s algorithm?
Introduction

Bregman Distance

Iterative Projection Algorithms

Belief Propagation

Conclusion