Coding Perspectives for Collaborative Estimation over Networks

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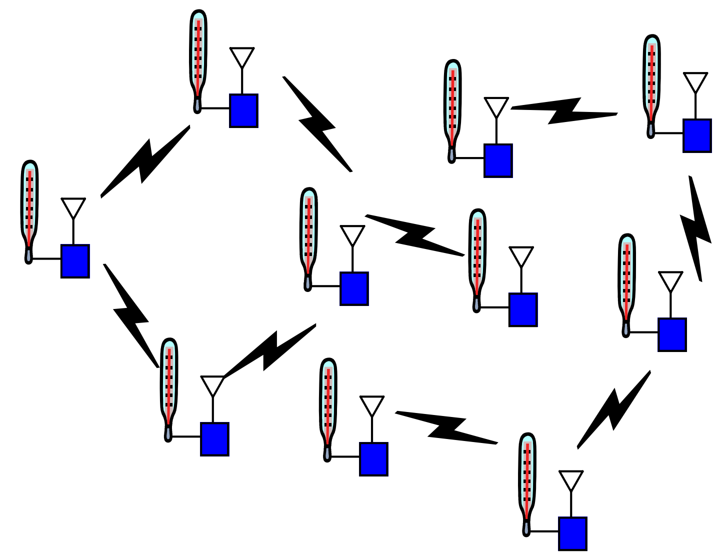
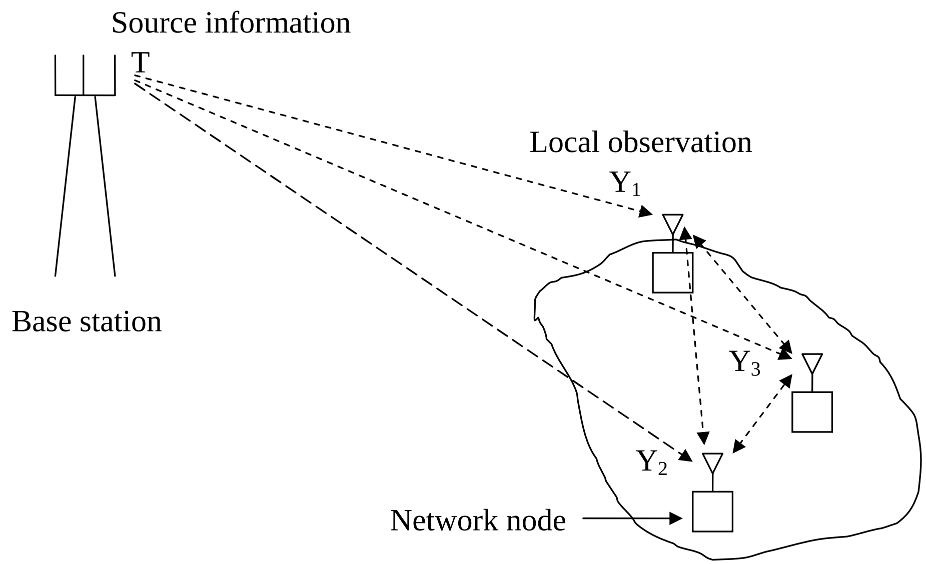
Outline

1. What is collaborative estimation? What are the major research issues/perspectives?
2. Collaborative Estimation from a Signal Processing/Machine Learning perspective
   (a) variational inference for collaborative estimation in sensor networks
   (b) Underlying Information Geometric Fundamental Problem
   (c) example w/ benefits: channel gain estimation
3. Collaborative Estimation from an Information Theory/Coding Perspective
   (a) proper architecture for the lossy network source code
   (b) related known lossy source coding problems
   (c) inner and outer bounds on the rate distortion region
   (d) Underlying Entropy Geometric Fundamental Problem
4. How might these perspectives be reconciled?

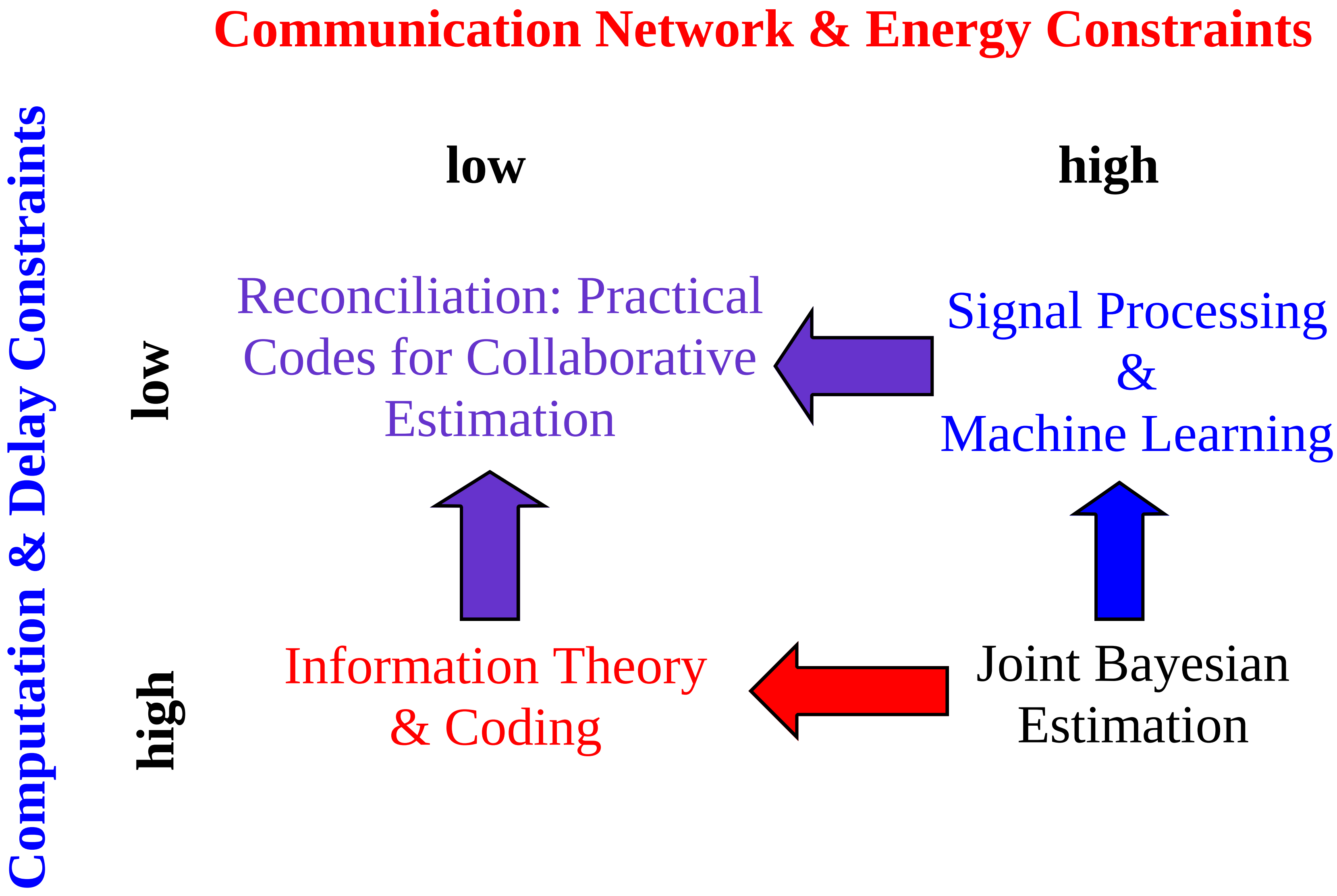
What is collaborative estimation?

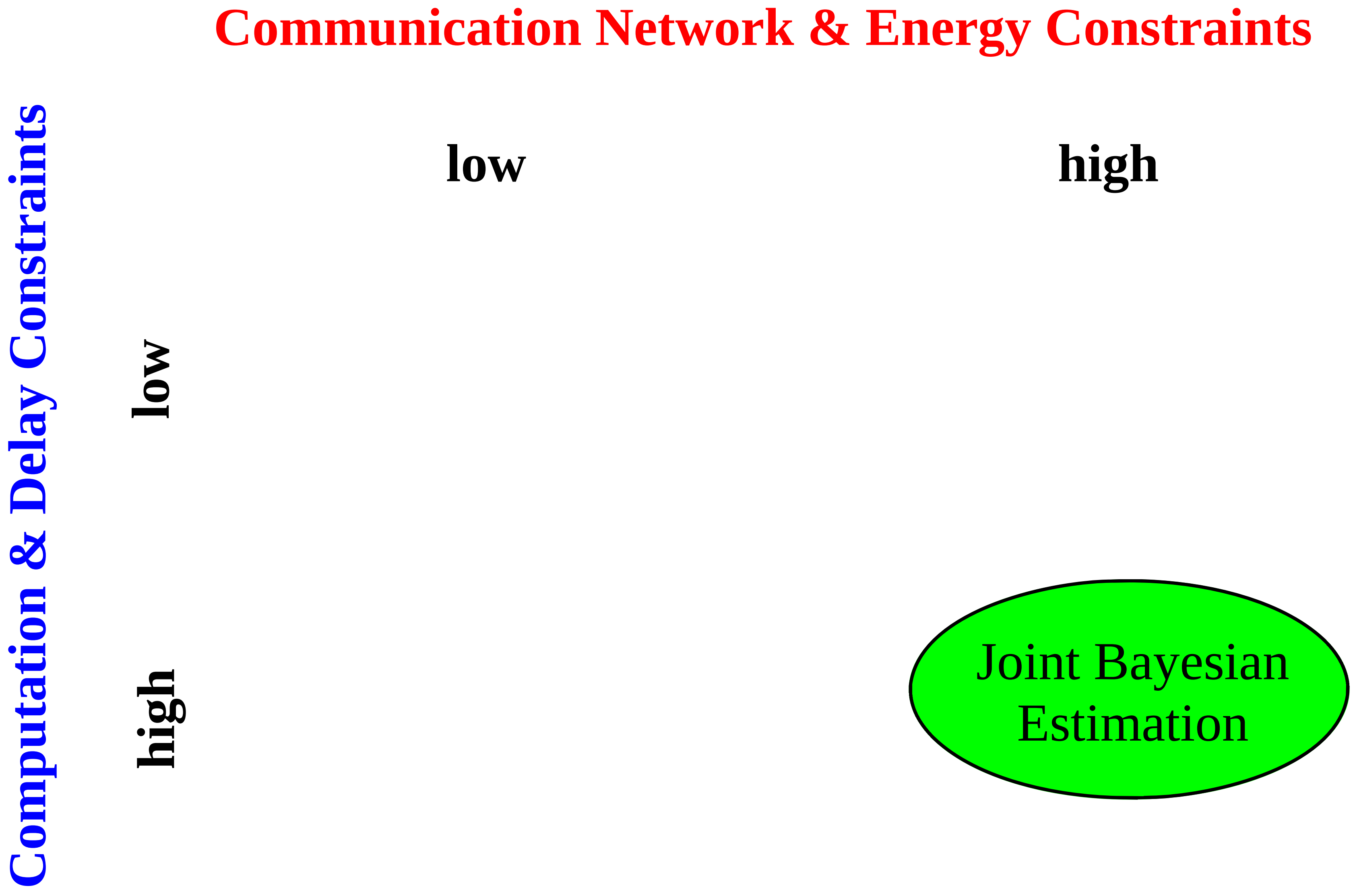




- M nodes. Node m w/ local observations $\mathbf{R}_m$.
- Collection of random parameters $\boldsymbol{T}$ jointly distrib. w/ $\{\mathbf{R}_m\}$
- Node m wants to estimate $\boldsymbol{T}$ with $\hat{T}_m$ to minimize a local Bayesian cost function, i.e. $d_m(\hat{T}_m, \boldsymbol{T})$ given avail. info.
- nodes share information over a network to help form their estimates

What are the major research issues/perspectives?




Communication Network & Energy Constraints
low
high
Computation & Delay Constraints
low
high
Joint Bayesian
Estimation

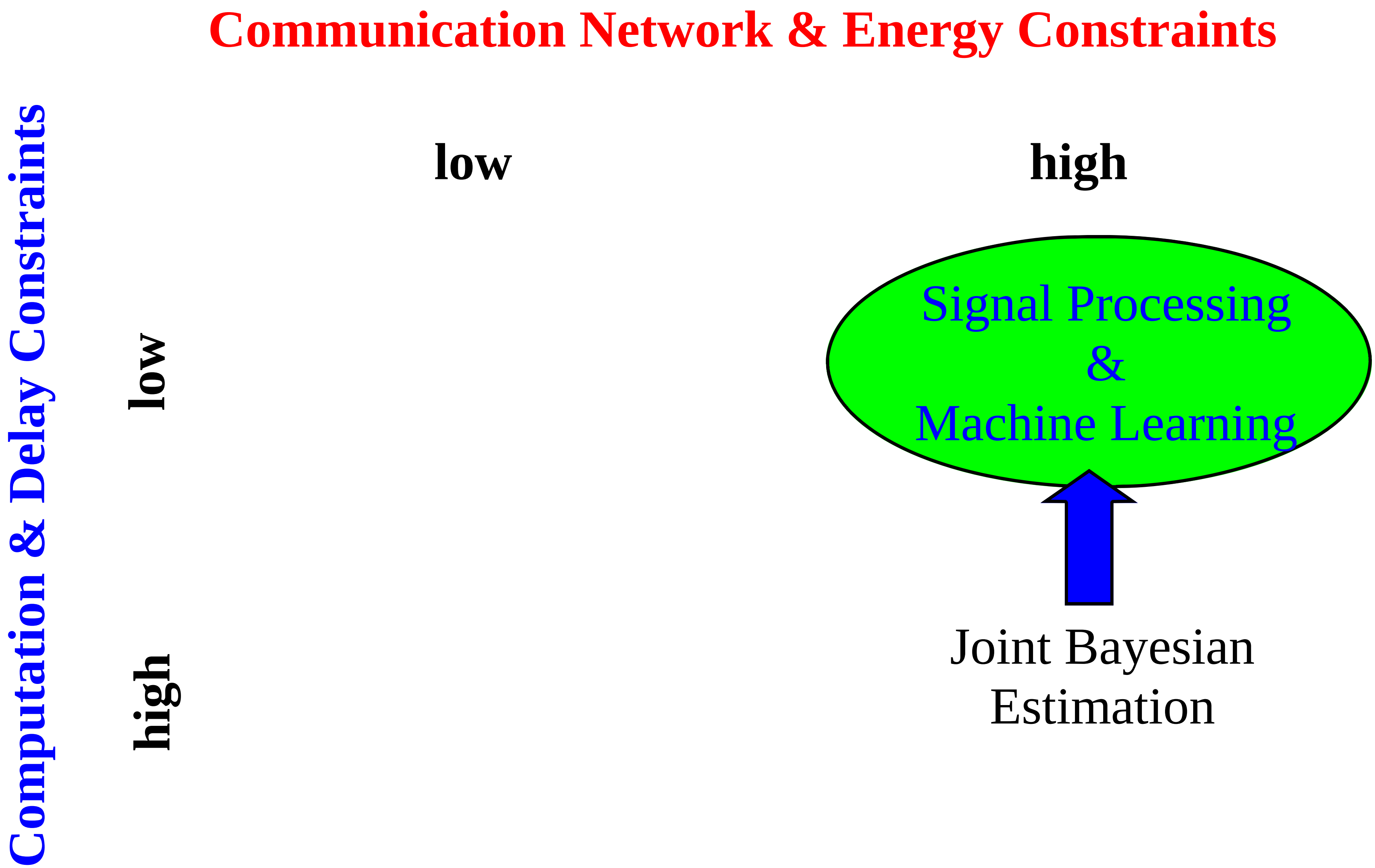
Joint Bayesian Estimation

- *Without the constraints, the problem is trivial once the model has been selected.*
- each node broadcasts its observations $\mathbf{r}_m$ to all of the other nodes
- given $\mathbf{r} := [\mathbf{r}_m | m \in [M]]$ each node forms the posterior distribution $p_{\mathbf{T}|\mathbf{R}}(\mathbf{T}|\mathbf{r})$.
- each node chooses its estimate $\hat{\mathbf{T}}_m$ as the estimate minimizing its own Bayesian risk function

$$\boxed{\hat{\mathbf{T}}_m \in \arg\min_{\hat{\mathbf{T}}_m} \int d_m(\hat{\mathbf{T}}_m, \mathbf{T}) p_{\mathbf{T}|\mathbf{R}}(\mathbf{T}|\mathbf{r}) \mathrm{d}\mathbf{T}}$$

Communication Network & Energy Constraints

Computation & Delay Constraints		**low**	**high**
	low		
	high		Joint Bayesian Estimation

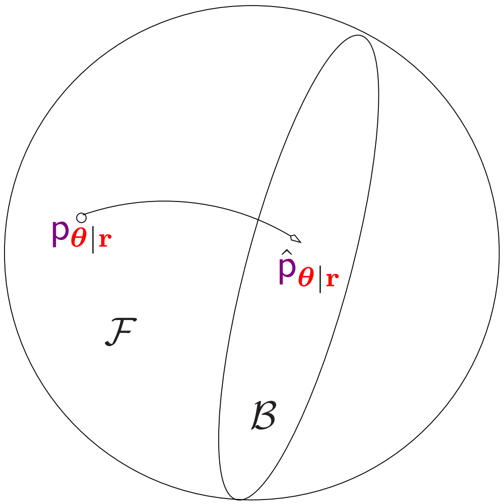

Communication Network & Energy Constraints
low
high
Computation & Delay Constraints
low
high
Signal Processing
&
Machine Learning
Joint Bayesian
Estimation

The Signal Processing/Machine Learning Perspective

PROBLEM 1: $\arg\min_{\hat{\mathbf{T}}_m} \int d_m(\hat{\mathbf{T}}_m, \mathbf{T}) p_{\mathbf{T},\mathbf{R}}(\mathbf{T}, \mathbf{r}) \mathrm{d}\mathbf{T}$ IS HARD!

- one important major difficulty: the integration over the posterior distribution is usually difficult computationally and analytically, as can be the minimization of the local risk.

  - enter approximate Bayesian inference: approximate the posterior distribution within a tractable family of distributions
    - Gibbs Sampling
    - Variational Bayes
    - Expectation/Belief Propagation



- Complexity handled in 2 respects
  - Approximating Family selected so that risk calc. & min. is easy
  - a factoring of $p(\boldsymbol{T}|\boldsymbol{r}) = \prod_a f_a(\boldsymbol{T}_a, \mathbf{r}_a)$ is exploited to individually fit factors of the approximate distribution (yields a "message passing" interpretation)

How can this be used to simplify the Risk Minimization/Calculation?

$$\arg\min_{\hat{\mathbf{T}}_m} \int d_m(\hat{\mathbf{T}}_m, \mathbf{T}) \hat{p}_{\mathbf{T}|\mathbf{R}}(\mathbf{T}|\mathbf{r}) \mathrm{d}\mathbf{T}$$

- if $d_m(\hat{\mathbf{T}}_m, \hat{T}) = d_m(\hat{\mathbf{T}}_m, \hat{T}_m)$ can select a factoring & approximating family to get marginals for $\mathbf{T}_m$.
- If the risk is a sum of terms of this form, can again simply find the best marginal approx., i.e. $\hat{p}(\mathbf{T}|\mathbf{r}) = \prod_m \hat{p}(\mathbf{T}_M|\mathbf{r})$
- More broadly, if there are parts of the posterior which yield risk computation difficult, they can be approximated with exponential families in which it is simple (e.g. Gaussians). [8, 9]
- The message passing nature of the algorithm describes one way to handle decentralization of the data (group it with factor nodes). [12, 13, 14, 15, 16]

What is the major underlying fundamental (math) problem here?

- The selection of the factoring

$$p(\boldsymbol{T}|\boldsymbol{r}) = \prod_a f_a(\boldsymbol{T}_a, \mathbf{r}_a) \quad \& \text{ the approximating family } \mathcal{B}$$

  determines *both*:

  - the convergence & the **complexity** of the variational inference, as well as
  - the **performance** of the estimates (i.e. the error in the risk calculation)

- Further, there is a tension:

  - A bigger approximating family allows for equal or better *performance*, but comes at the cost of additional *complexity* in fitting the approximate distribution and calculating the risk.
  - Additionally, a bigger approximating family requires more parameters, and hence requires bigger messages, hence more communication.

- *Characterize this performance vs complexity tradeoff and approximating families which attain it*. (Dictated by interplay between parameterizations of subfamilies of distributions and estimate errors, i.e. information geometry.)

Expectation Propagation (EP), I [1, 2, 3]

- parameters $\underline{\mathbf{T}}$ whose a.p.d.'s we want
- observations $\underline{\mathbf{r}}$
- joint stat. model that factors $\mathbf{T}_a \subseteq \mathbf{T}$

$$\mathsf{p}_{\underline{\mathbf{r}},\underline{\mathbf{T}}}(\mathbf{r},\mathbf{T}) \propto \prod_{a=1}^{M} \mathsf{f}_{a,\mathbf{r}}(\mathbf{T}_a)$$

- Goal: calculate $\boldsymbol{\lambda}_a(\mathbf{r})$ to approximate

$$\mathsf{p}_{\underline{\mathbf{T}}|\underline{\mathbf{r}}}(\mathbf{T}|\mathbf{r}) \approx \prod_{a=1}^{M} \mathsf{g}_{a,\boldsymbol{\lambda}_a(\mathbf{r})}(\mathbf{T}_a)$$

- $\mathsf{g}_{a,\boldsymbol{\lambda}_a(\mathbf{r})}(\mathbf{T}_a) \propto \exp\left(\mathbf{h}_a(\mathbf{T}_a)\cdot\boldsymbol{\lambda}_a(\mathbf{r})\right)$
- Designer selects $\mathbf{h}(\cdot) := [\mathbf{h}_a(\cdot)]$
- Given design + $\mathbf{r}$, EP $\to \boldsymbol{\lambda}_a(\mathbf{r})\ \forall a$.

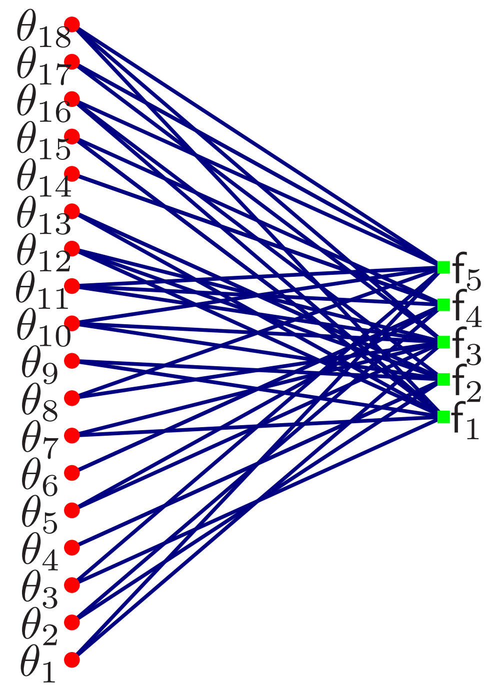


Figure 1: The parameter factor graph.

EP, II. Design Choices: The log-basis Functions $\mathbf{h}(\mathbf{T})$

- In choosing $\mathbf{h}(\cdot)$ one trades between
  1. Accuracy: level of stat. dep. amongst T_i allowed in approx.
  2. Complexity: control amnt of computation + comm. req.'d
- Requirement: Sufficiency $\forall \mathbf{T}_a$

$$\mathsf{f}_a(\mathbf{T}_a) = \hat{\mathsf{f}}_a(\mathbf{h}_a(\mathbf{T}_a))$$

  so all information f_a depends on is in $\mathbf{h}_a(\mathbf{T}_a)$.
- Requirement: Reciprocity $\mathbf{h}_a$s are concatenations of $\mathbf{v}_i(\mathbf{T}_i)$ with each T_j in only one $\mathbf{T}_i$.
- $\implies$ everywhere we are approximating T_i we are using the same type of density.

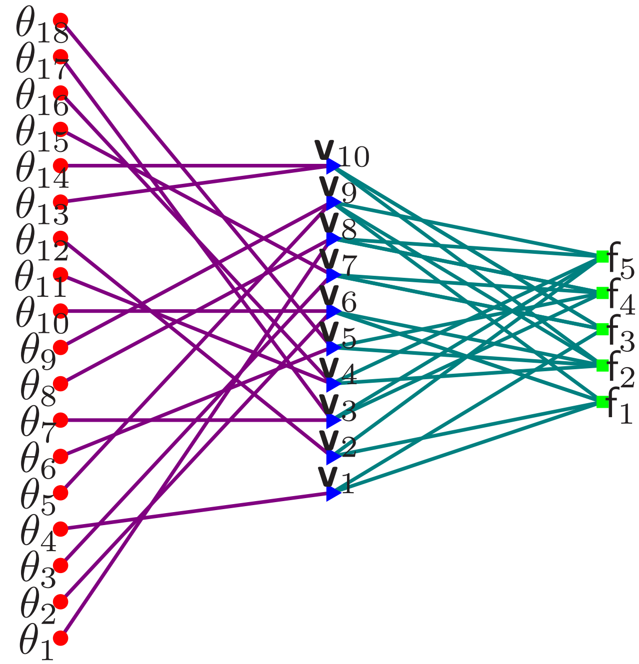


Figure 2: The parameter statistics factor graph.

EP, III. Model Selection via $\mathbf{h}(\mathbf{T})$: Examples

Discrete Θ (recall $\underline{\mathbf{T}} \in \Theta$): Consider $\Theta = \{0,1\}^N$

- **independent bits:** $\mathbf{h}(\mathbf{T}) = \mathbf{T}$
- **pairwise dependent bits:**

$$\mathbf{h}(\mathbf{T}) = [T_1, \cdots, T_N, T_1T_2, T_1T_3, \cdots, T_1T_N, T_2T_3, \cdots, T_2T_N, \cdots, T_{N-1}T_N]^T$$

Continuous Θ: Consider $\Theta = \mathbb{R}^N$

- $\{T_i\}$ **independent Gaussian:**

$$\mathbf{h}(\mathbf{T}) := \left[T_1, T_1^2, T_2, T_2^2, \cdots, T_N, T_N^2\right]^T$$

- $\{T_i\}$ **jointly normal:**

$$\mathbf{h}(\mathbf{T}) := \left[T_1, T_1^2, T_2, T_2^2, \cdots, T_N, T_N^2, T_1T_2, T_1T_3, \cdots, T_1T_N, T_2T_3, \cdots, T_2T_N, \cdots, T_{N-1}\right.$$

Other possible distribution types: exponential, beta, gamma, Poisson, any finite distribution

EP, IV: The Message Passing Algorithm

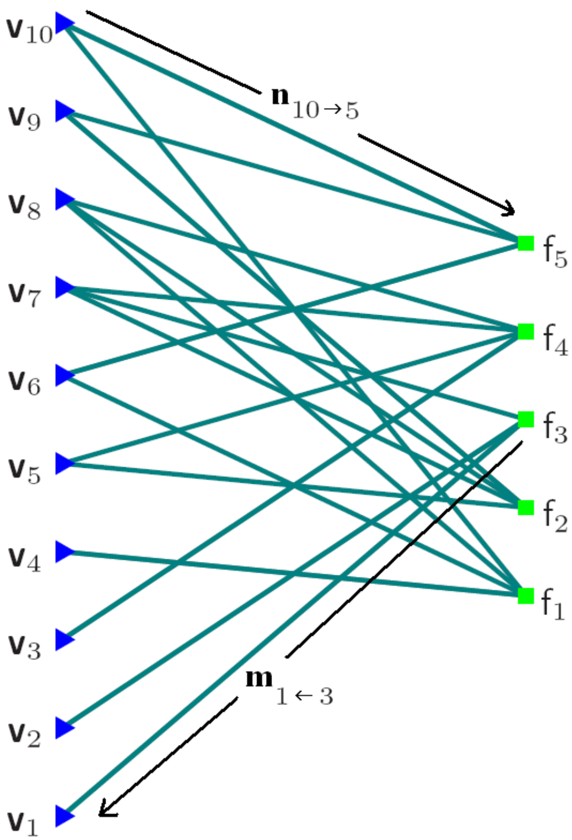


Figure 3: Message Passing.

- Try to make the approx. by passing messages
- right going messages

$$[\boldsymbol{\lambda}_{\text{in}}]_i := \sum_{c \in \mathcal{N}(i)\setminus\{a\}} [\boldsymbol{\lambda}_c]_i =: [\mathbf{n}_{j\to a}]_i$$

- left going messages

$$\mathbf{m}_{j\leftarrow a} := [[\boldsymbol{\lambda}_a]_i \,|\mathrm{h}_i \in \mathbf{v}_j]$$

$$\frac{\int_{\Theta_a} \mathbf{h}_a(\mathbf{T}_a)\hat{\mathrm{f}}_a(\mathbf{h}_a(\mathbf{T}_a)) \exp\left(\mathbf{h}_a(\mathbf{T}_a)\cdot\boldsymbol{\lambda}_{\text{in}}\right) \mathrm{d}\mathbf{T}_a}{\int_{\Theta_a} \hat{\mathrm{f}}_a(\mathbf{h}_a(\mathbf{T}_a)) \exp\left(\mathbf{h}_a(\mathbf{T}_a)\cdot\boldsymbol{\lambda}_{\text{in}}\right) \mathrm{d}\mathbf{T}_a}$$

$$= \frac{\int_{\Theta_a} \mathbf{h}_a(\mathbf{T}_a) \exp\left(\mathbf{h}_a(\mathbf{T}_a)\cdot(\boldsymbol{\lambda}_{\text{in}}+\boldsymbol{\lambda}_a)\right) \mathrm{d}\mathbf{T}_a}{\int_{\Theta_a} \exp\left(\mathbf{h}_a(\mathbf{T}_a)\cdot(\boldsymbol{\lambda}_{\text{in}}+\boldsymbol{\lambda}_a)\right) \mathrm{d}\mathbf{T}_a}$$

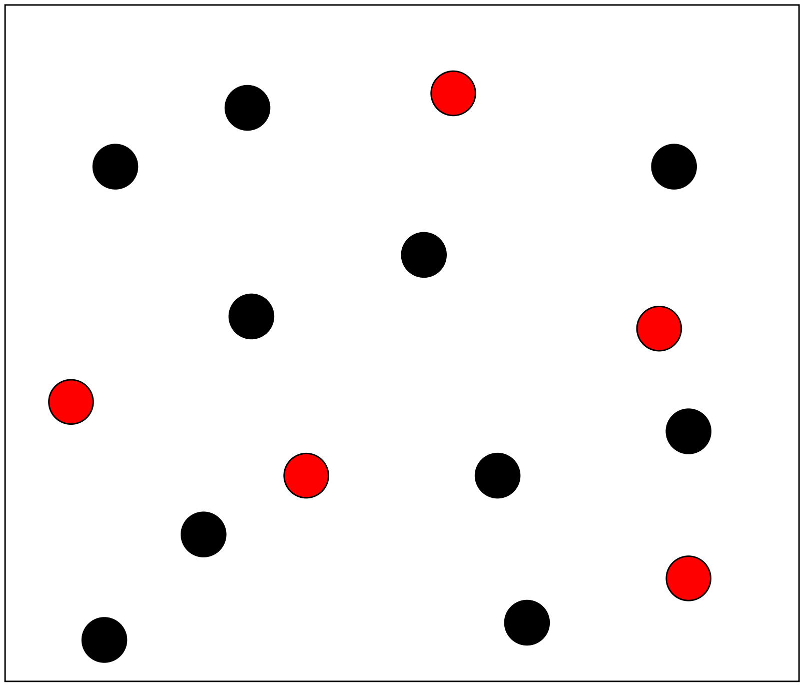
When can you prove the approximation is good?

- When the factor node updates are equivalent to passing parameters for the associated marginal density, EP $=$ belief propagation.[4, 5]
- In this case if the factor graph is a cycle free, BP gives exact marginal distributions for the (clustered) Θ_i
- Since it is a local message passing algorithm, if a particular computation neighborhood of size 2ℓ is a tree, then an exact posterior is calculated over data available in that neighborhood. [6, 7]
- Factoring can be set up to give tree like neighborhoods via random duty cycling, as shown in the next example. [10, 11]

Example: Wireless Sensor Network Initialization

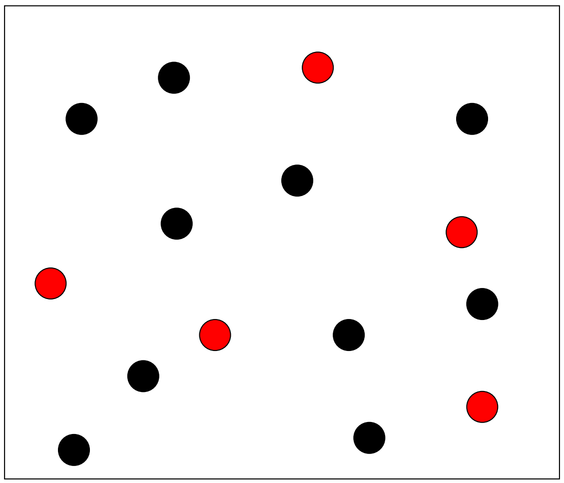
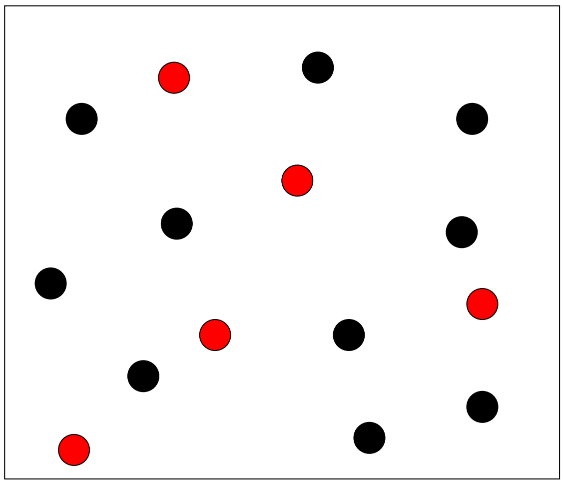
- Plane flies over forest/ field of interest, drops many sensor nodes.
- Placement is random, nodes do not know: positions, their neighbors, nor the gains in the wireless channels between them.
- Nodes must conserve power $\Longrightarrow$ duty cycling a must, but can not yet be done in an organized fashion.
- Organization performed in network, egalitarian (non-hierarchical) structure.
- Each node would like to organize communications in an energy efficient manner (power control, MAC, and routing), but this requires knowledge of channel gains.
- Initialization Phase: Using a random sleep (duty cycling) strategy for communications, estimate the wireless channel gains in the network.

Duty cycling to the network



- Keep only a small subset of sensors "awake" at each time instant

Regular cyclic random sleep strategy to the network

- Regular cyclic random sleep strategy: a random subset of nodes are awake at a time and the sleep pattern repeats after certain amount of time
- At each time instant same number of nodes are awake and each maintains the same average power consumption

Regular cyclic random sleep strategy to the network

- d nodes are awake at a time instant
- Each node is awake c times in a sleep cycle
- K time instants in a sleep cycle

$$K = \frac{c}{d} N$$

- Define the set of nodes awake at time instant k to be $\{\mathcal{S}(k) | k \in \{1, \ldots, K\}\}$

Model for the channels

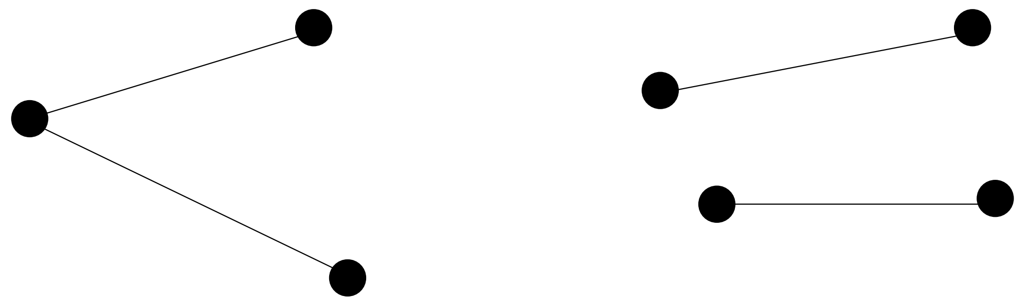
- Channel gain of a link between any two nodes heavily depends on the distance between them
- Pathloss model

$$h \propto R^{-n}$$

  where pathloss exponent n $(2 \leq n \leq 6)$
- Gain of the link between node i and node j

$$h_{i,j} \propto \|\mathbf{x}_i - \mathbf{x}_j\|_2^{-4}$$

The prior joint distribution of the channels



- Any two channel gains incident on the same node are dependent

$$\text{dependent:} \qquad h_{i,j} \propto \|\mathbf{x}_i - \mathbf{x}_j\|_2^{-4} \qquad h_{i,m} \propto \|\mathbf{x}_i - \mathbf{x}_m\|_2^{-4}$$

$$\text{independent:} \qquad h_{i,j} \propto \|\mathbf{x}_i - \mathbf{x}_j\|_2^{-4} \qquad h_{m,n} \propto \|\mathbf{x}_m - \mathbf{x}_n\|_2^{-4}$$

- The prior joint distribution of the channel gains is analytically complex, because of the inverse nonlinear dependence on the node positions
- Under EP, we approximate it with a Gaussian with the same mean and covariance
- Ability to exploit this prior information is key from a *network* perspective.

Channel Training

- Train the channel using a training sequence $u_1, \ldots, u_M$
- Model the observation as

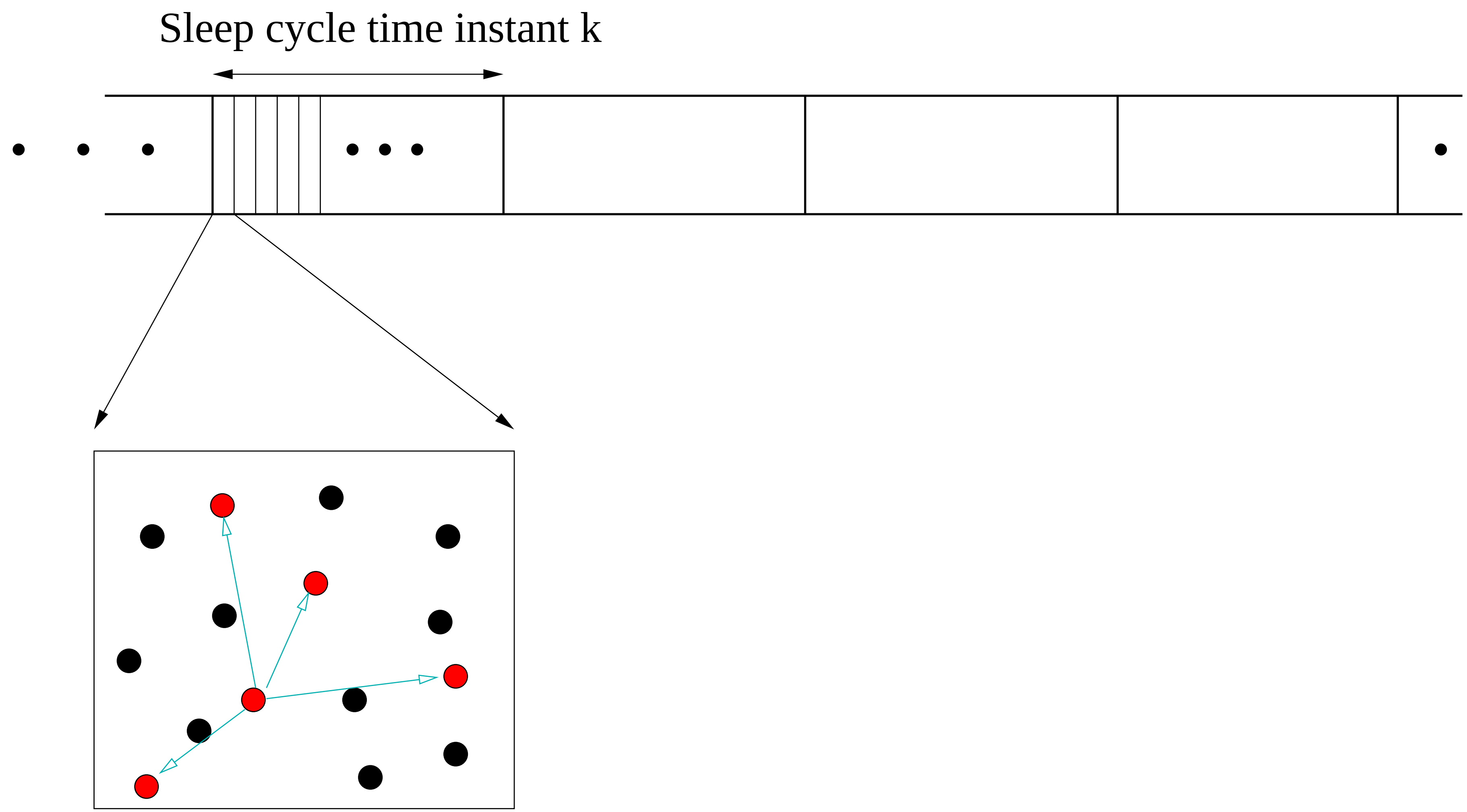
$$r_m = hu_m + v_m$$

where v_m is Gaussian distributed noise, which is i.i.d. over time and space

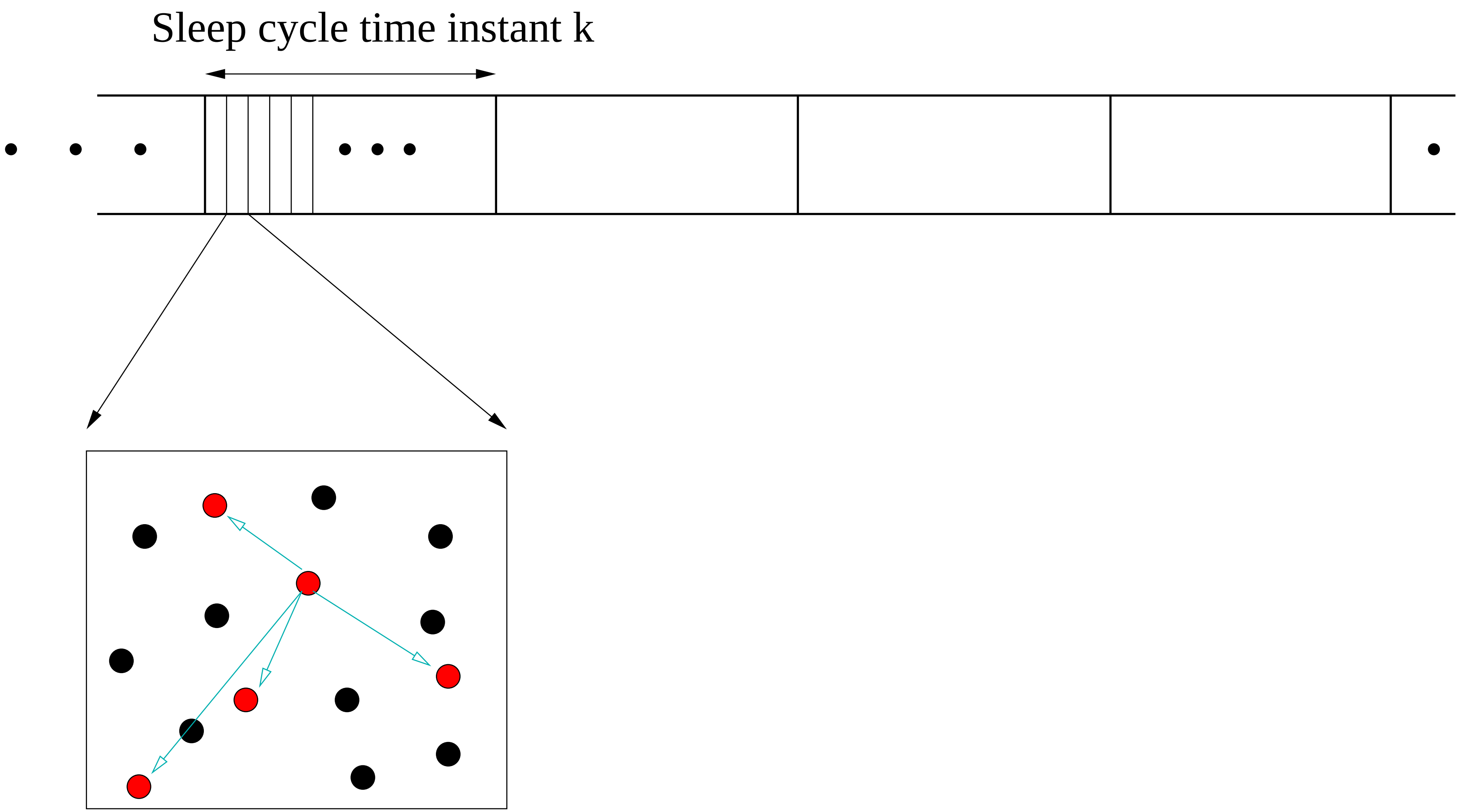
- Each time instant k is further divided into $2c$ time slots
- In the first c time slots, nodes which are awake during k take turns transmitting their training sequences

$$r_{k,i,j,m} = h_{i,j} u_{i,m} + v_{k,i,j,m}$$

Channel Training



Channel Training



- Collect the observations during sleep cycle time instant k

$$\mathbf{r}_k := [r_{k,i,j,m} \mid i,j \in \mathcal{S}(k), m \in \{1,\ldots,M\}, i \neq j]$$

The joint distribution of channel gains and observations

- The joint probability distribution of $\mathbf{r}$ and $\mathbf{h}$

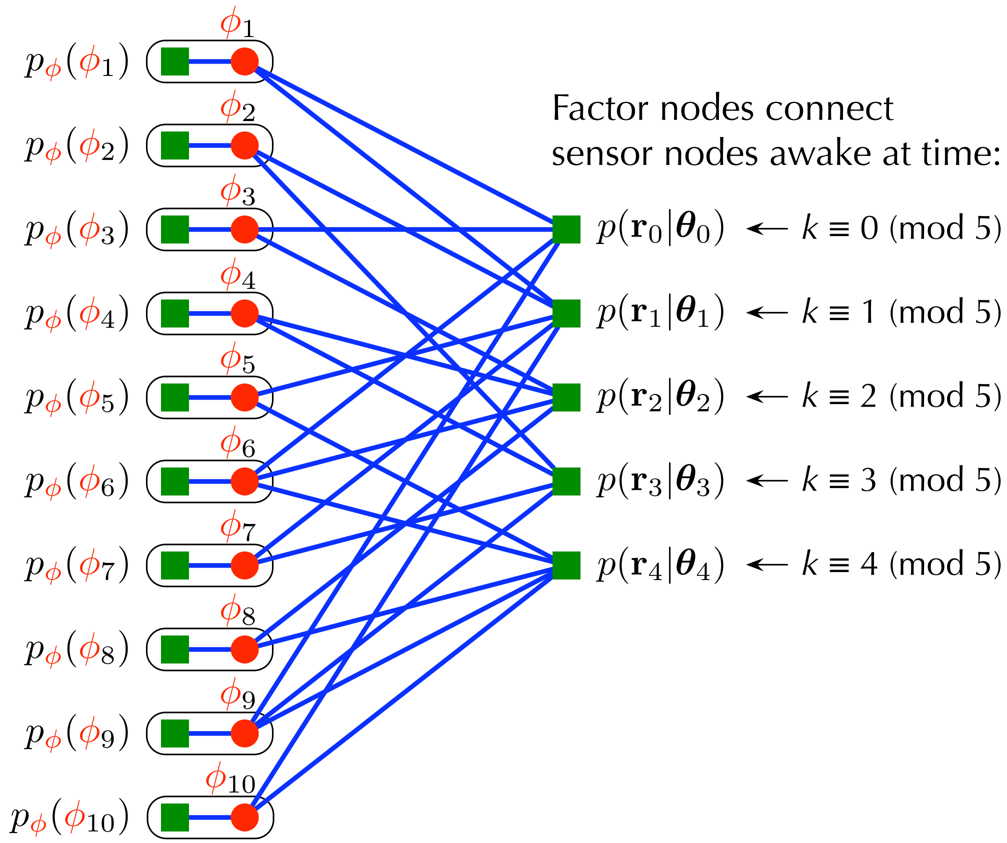
$$p_{\mathbf{r},\mathbf{h}} = p_{\mathbf{h}} \prod_{k=1}^{K} p_{\mathbf{r}_k|\mathbf{h}}$$

  where $\mathbf{r} := [\mathbf{r}_k \mid k \in \{1, \ldots, K\}]$
- Each node has a copy $\mathbf{h}_i$ of $\mathbf{h}$
- Write the joint distribution as

$$p_{\mathbf{r},\mathbf{h},\mathbf{h}_1,\ldots,\mathbf{h}_N} = \prod_{k=1}^{K} p_{\mathbf{r}_k|\mathbf{h}} \prod_{i=1}^{N} \delta(\mathbf{h} - \mathbf{h}_i)(p_{\mathbf{h}}(\mathbf{h}_i))^{\frac{1}{N}} \tag{1}$$

  where δ is the point mass distribution at zero.
- We can associate this model with a factor graph

Factor graph

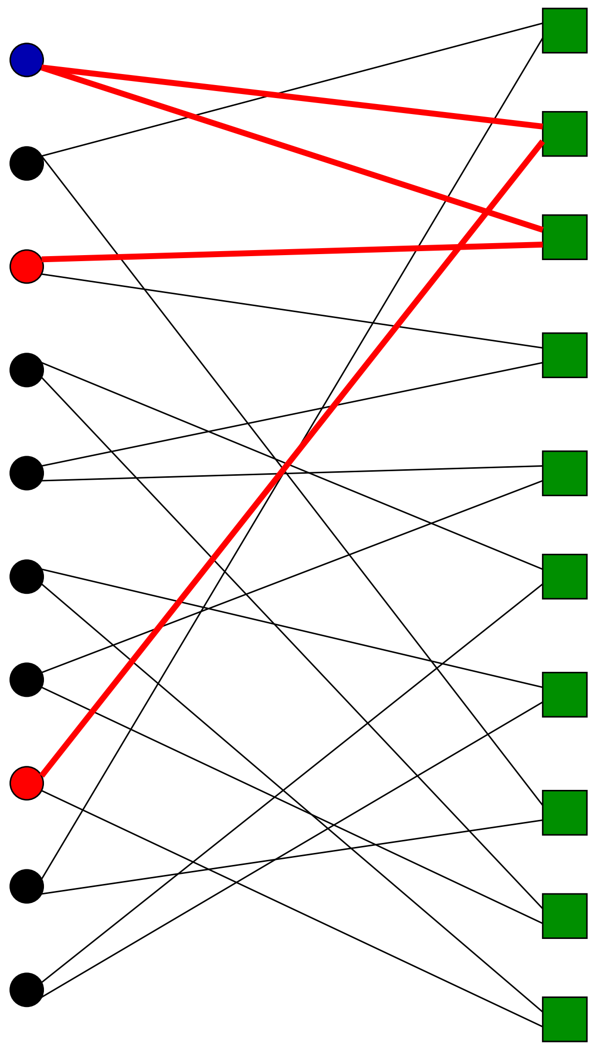


- A bipartite graph
- Left nodes: variable nodes, Right nodes: factor nodes
- Represent sensor nodes with the variable nodes and sleep cycle time instant with the factor nodes

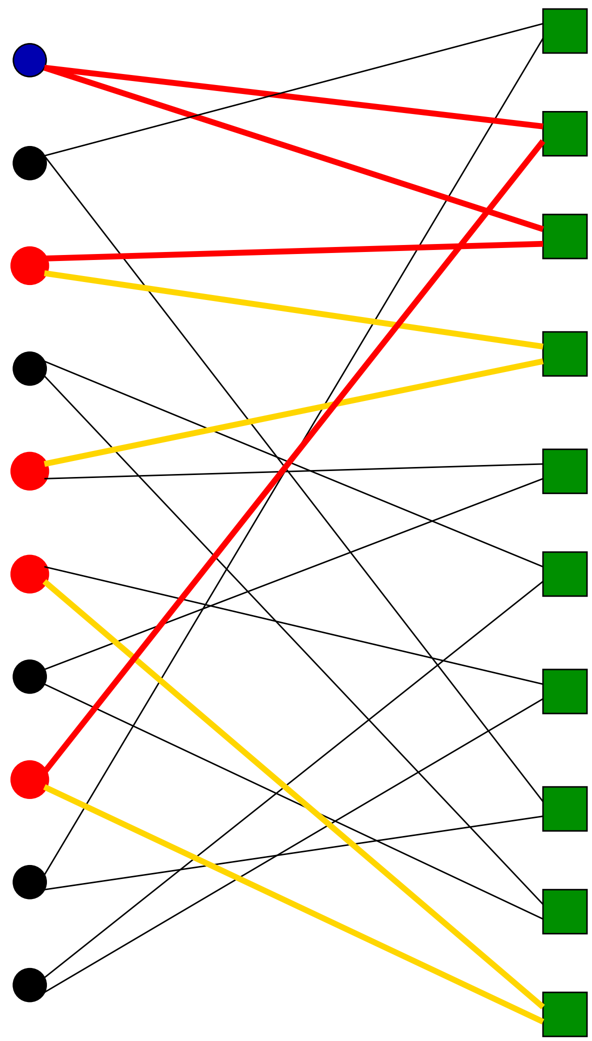
The channels observed by a node after ℓ sleep cycles

- Only a subset of the channels can be observed after ℓ sleep cycles
- Assumption: a node cannot disseminate data received during time instant k at another time instant k' in the same sleep cycle
- Decoding of data from other nodes takes time on the order of one complete sleep cycle
- After ℓ sleep cycles, nodes can directly or indirectly receive information about links observed by nodes only upto 2ℓ edges away from them in the factor graph.

The channels observed by a node after 1 sleep cycle



The channels observed by a node after 2 sleep cycles



Expectation Propagation

- The approximated prior joint distribution on $\mathbf{h}$ can be written as

$$p_{\mathbf{h}}(\mathbf{h}) \propto \exp\{-\frac{1}{2}[(\mathbf{h}-\mathbf{m_h})^T \mathbf{\Sigma}_{\mathbf{h}}^{-1}(\mathbf{h}-\mathbf{m_h})]\} \tag{2}$$

- Initial estimate at each node $\hat{\mathbf{h}} = \mathbf{m_h}$
- They may want to update their estimates by updating the statistics (mean and covariance)
- Once we have associated the joint distribution on $\mathbf{h}$ with a factor graph, we can apply Expectation Propagation to calculate the posterior distribution

Selection of message family

- The approximated prior joint distribution on $\mathbf{h}$

$$p_{\mathbf{h}}(\mathbf{h}) \propto \exp\{-\frac{1}{2}[(\mathbf{h}-\mathbf{m_h})^T \mathbf{\Sigma}_{\mathbf{h}}^{-1}(\mathbf{h}-\mathbf{m_h})]\} \tag{3}$$

- The conditional joint distribution on the observations $\mathbf{r}_k$ collected during sleep cycle instant k

$$p_{\mathbf{r}_k|\mathbf{h}_k}(\mathbf{r}_k|\mathbf{h_k}) \propto \exp\{-\frac{1}{2}[(\mathbf{r}_k-\mathbf{m}_{\mathbf{r}k})^T \mathbf{\Sigma}_{\mathbf{r}_k}^{-1}(\mathbf{r}_k-\mathbf{m}_{\mathbf{r}k})]\} \tag{4}$$

- Select the the message exponential family to be used in EP to be multivariate Gaussian distributed as

$$\mathbf{v}(\mathbf{h}) = \begin{pmatrix} \mathbf{h}_y & \mathbf{h}_z & \mathbf{h} \end{pmatrix}^T$$

$$\begin{aligned}
\mathbf{h}_y &:= [h_{i,j}^2 | i,j \in \{1,\ldots,N\}, i<j] \\
\mathbf{h}_z &:= [h_{i,j}h_{m,n} | i,j,m,n \in \{1,\ldots,N\}, i<j, m<n, m>i] \\
\mathbf{h} &:= [h_{i,j} | i,j \in \{1,\ldots,N\}, i<j]
\end{aligned}$$

Diffusion LMS [17]

- Least-Mean Squares (LMS) is a stochastic gradient-descent algorithm
- During sleep cycle time instant k, when node i transmits, the nodes $i' \in \mathcal{S}(k) \setminus i$ make observations
- Node i' has access to $\{u_{i,m}, r_{k,i,i',m}\}$ $u_{i,m}$: regression vector, $r_{k,i,i',m}$: desired signal
- Estimate $\hat{h}_{i,i'}^{k,m}$ of $h_{i,i'}$ at node i'

$$\hat{h}_{i,i'}^{k,m} = \hat{h}_{i,i'}^{k,m-1} + \mu u_{i,m}(r_{k,i,i',m} - \hat{h}_{i,i'}^{k,m} u_{i,m})$$

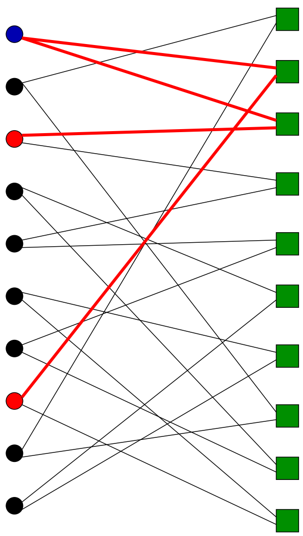
- At the end of the sleep cycle instant k, diffuse the estimate by

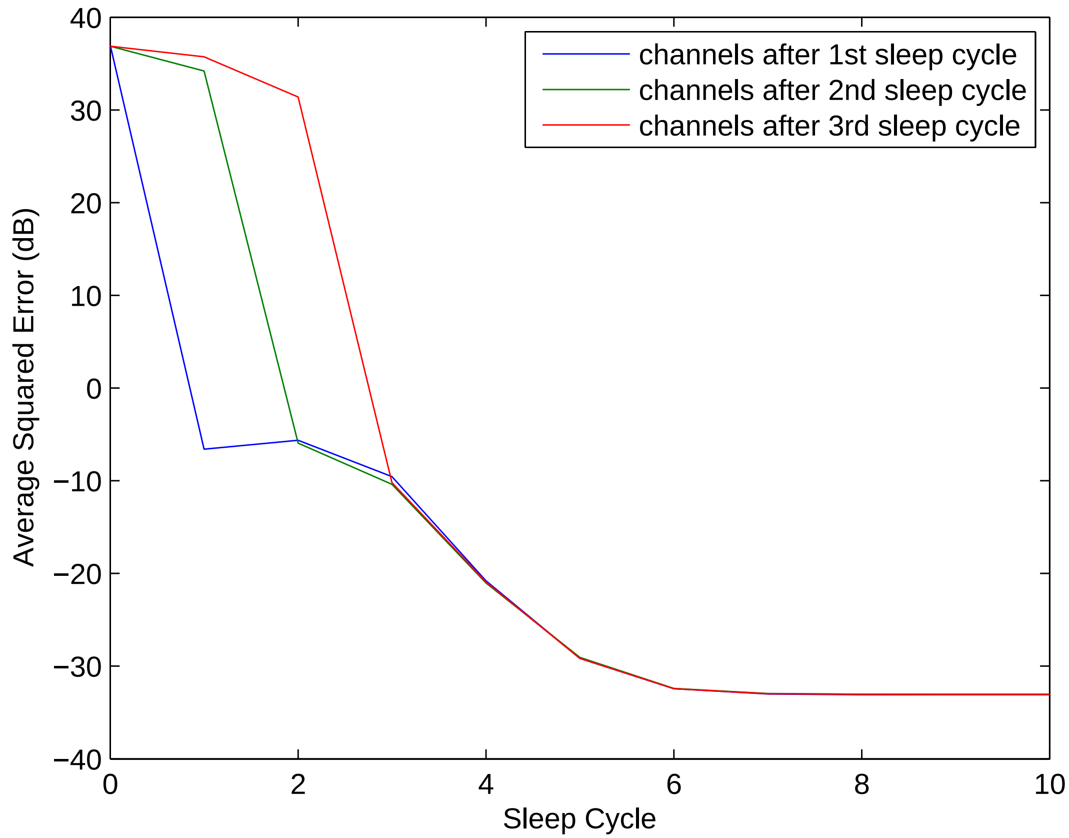
$$\tilde{\mathbf{h}}^k = \sum_{i \in \mathcal{S}(k)} a(k,i)\hat{\mathbf{h}}_i^k$$

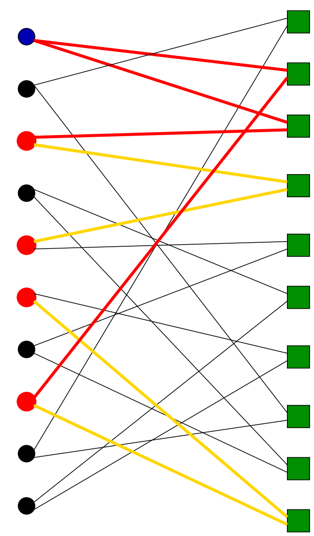
Simulation

- A network with 20 sensors
- Random sleep strategy with $K = 10, d = 4, c = 2$
- Training sequence of length 1000
- Monte Carlo Simulations 400
- Plot average estimation error of only those channel gains observed directly or indirectly after ℓ iterations

Simulation results: EP [8, 9]

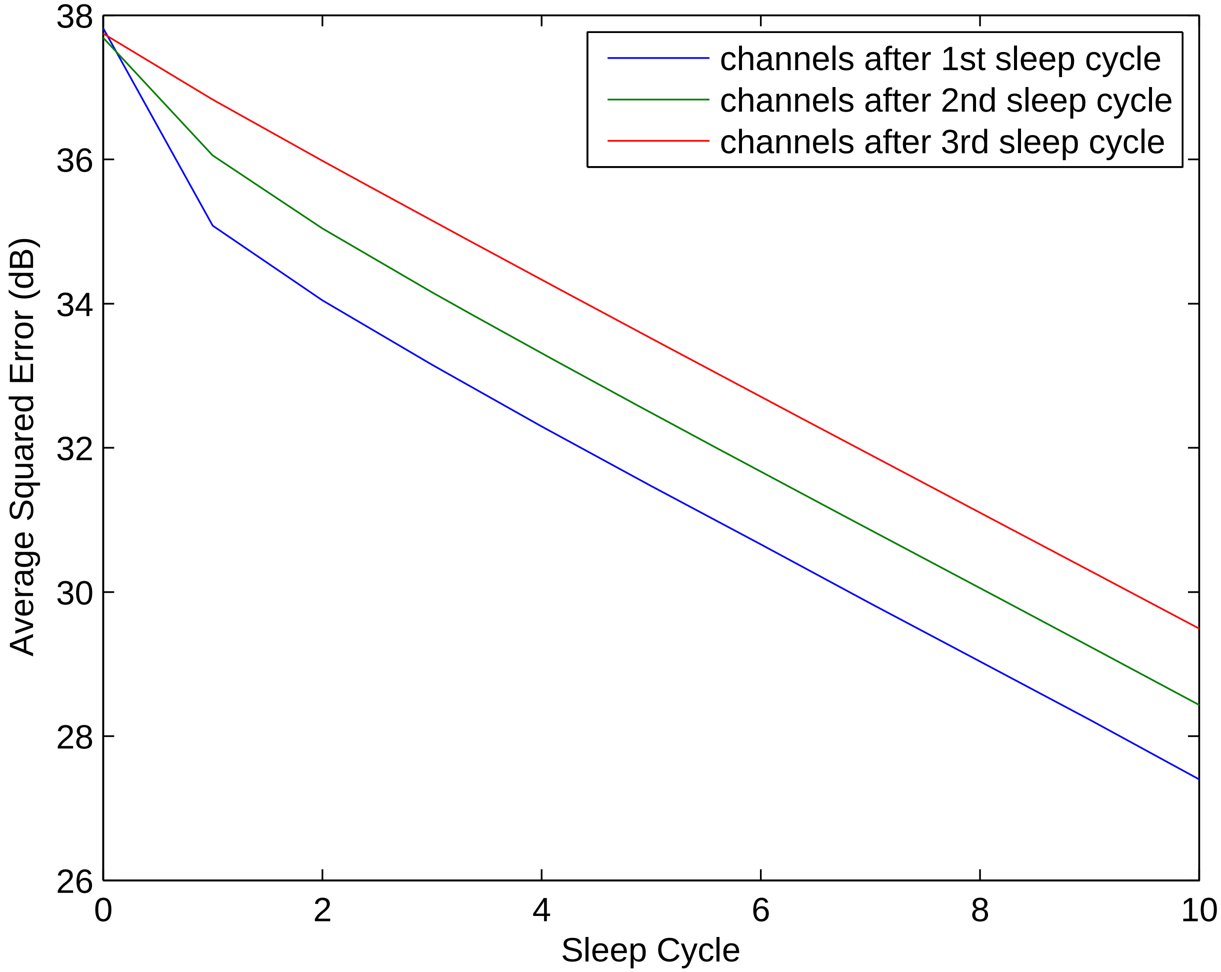






- For directly observed links, drastic change after first sleep cycle
- Drastic change shifts for the indirectly observed links
- Estimation error decreases even after that

Simulation results: LMS

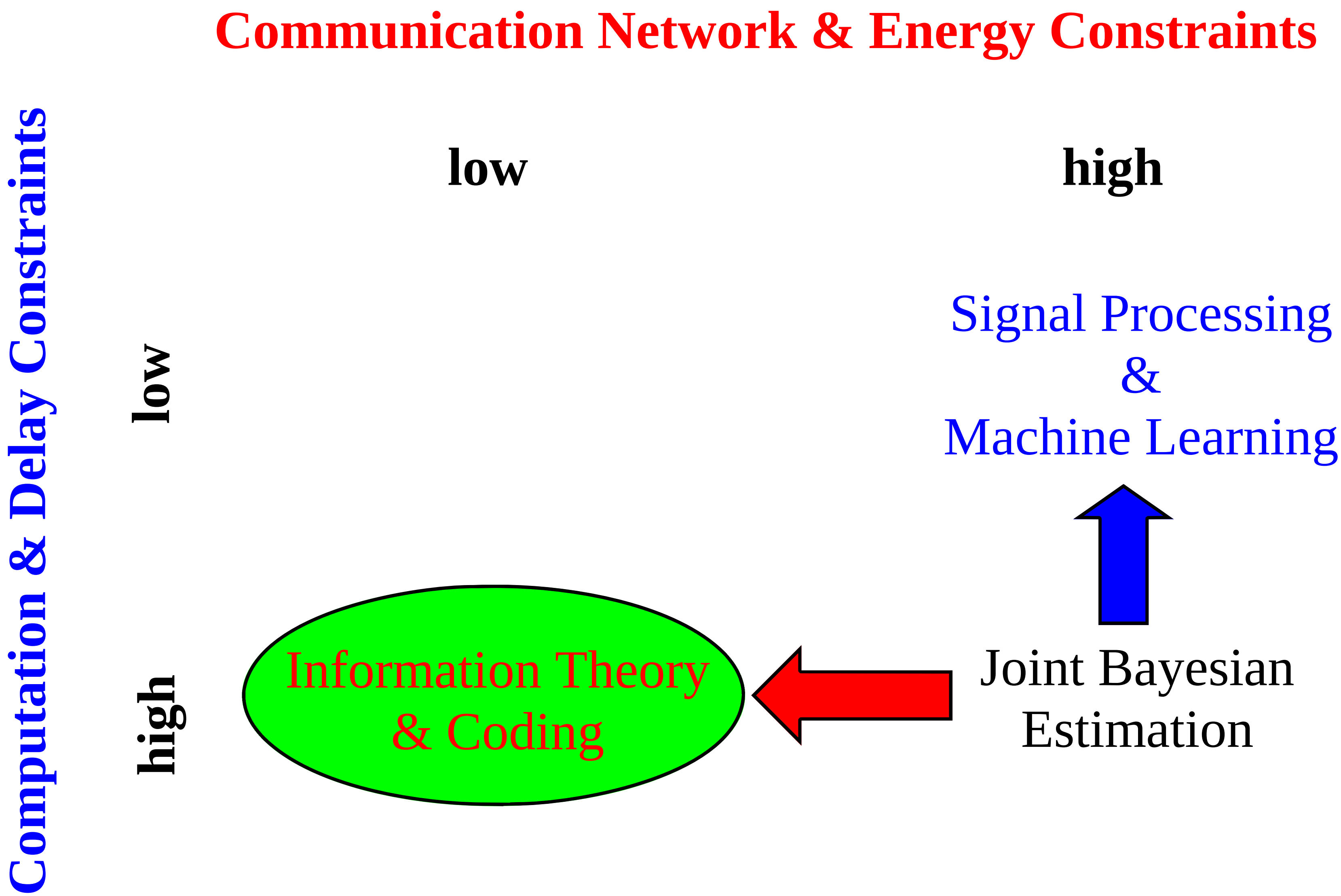


- Maximum step size before it starts to diverge: 1.995

Communication Network & Energy Constraints

Computation & Delay Constraints

	low	**high**
low		Signal Processing & Machine Learning
high		Joint Bayesian Estimation

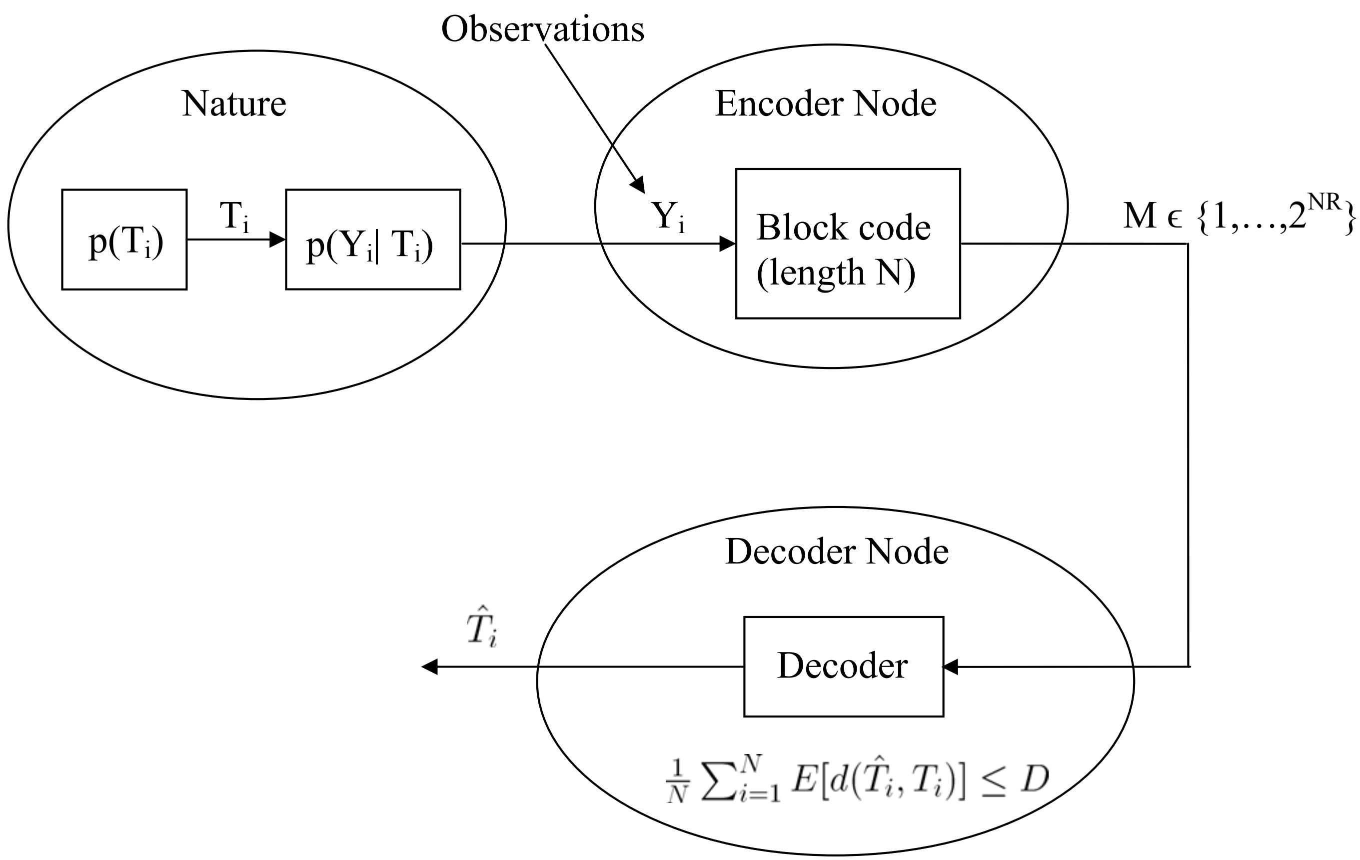

Communication Network & Energy Constraints
Computation & Delay Constraints
low
high
low
high
Signal Processing
&
Machine Learning
Information Theory
& Coding
Joint Bayesian
Estimation

The Information Theory/Coding Perspective

PROBLEM 2: only $\mathbf{r}_m$ is originally available to node m, any communication is over finite rate links

- another major important difficulty: the nodes must send their messages over a rate limited wired or wireless communication network. The information exchanged can not exceed the capabilities of this network.
- In a wireless network, the capabilities of the network are strongly related to the energy expenditures of the network nodes, due to a large amount of power spent on transmission.
- How should the communications be organized to allow for the best estimate performance when adapted to different communications networks? (I.e. what is the code structure?)
- What is the *best* estimate performance we can have subject to these constraints?

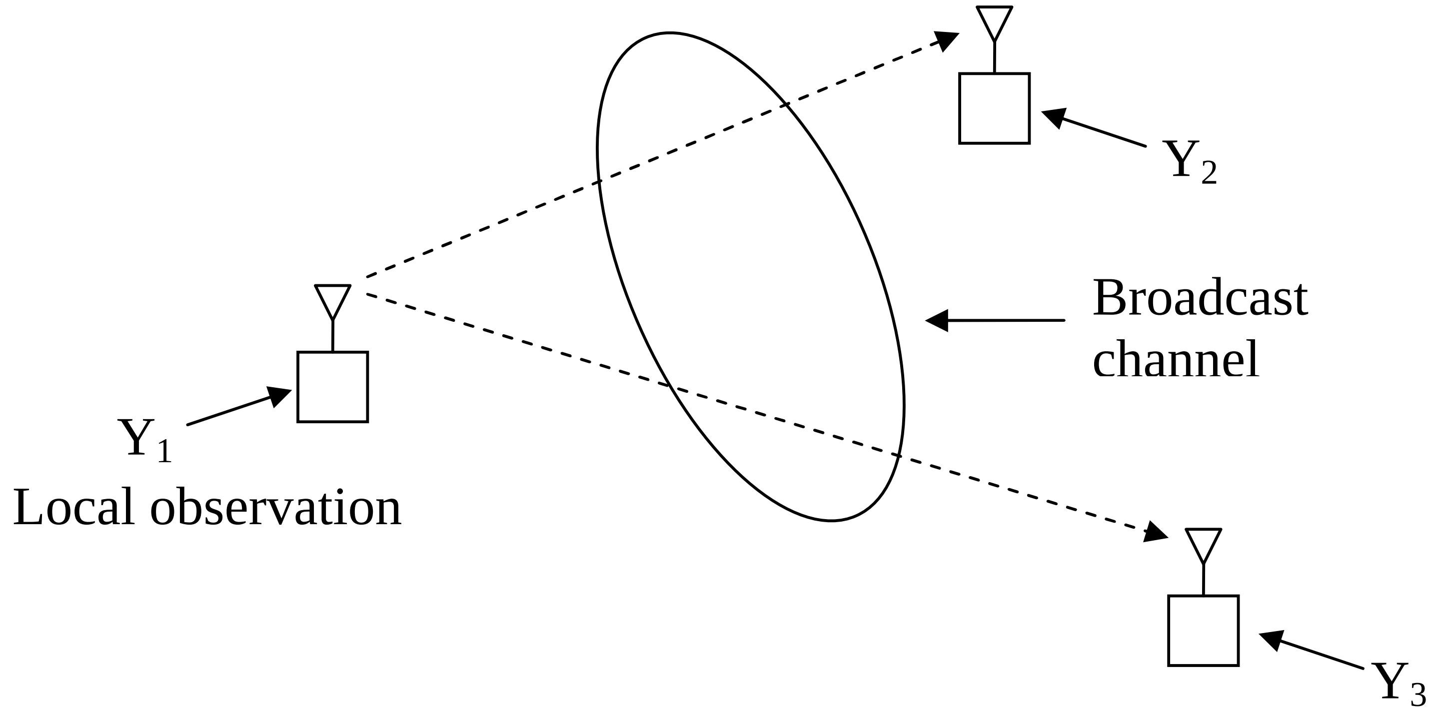
Relationship Between Remote Bayesian Estimation and Lossy Source Coding



- $\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}[d(\hat{T}_i, T_i)] < D$ plays the role of an average Bayesian cost. Dobrushin & Tsybakov '62 [18] showed minimum rate necessary to attain $< D$ is $\min I(U;Y)$ over $U \leftrightarrow Y \leftrightarrow T$.

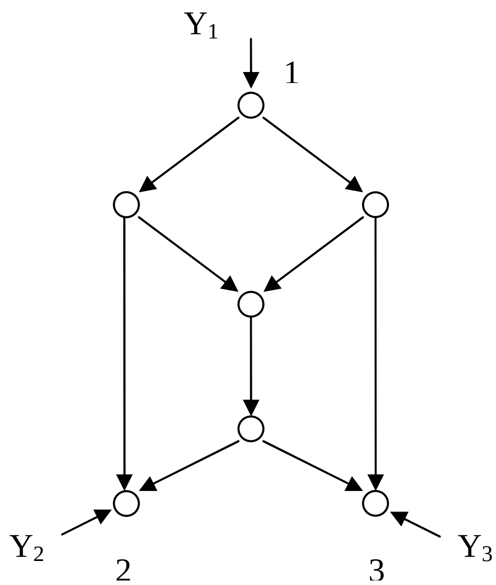
- Just like rate distortion function but with T_i instead of Y_i in distortion, and Markov requirement.

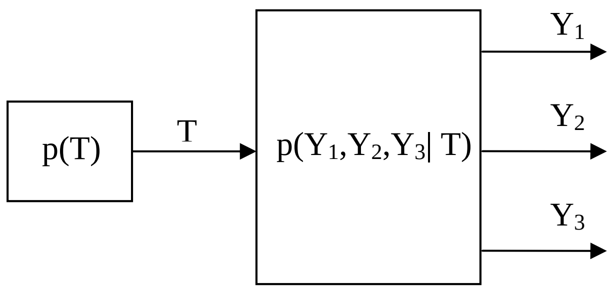
What should the source code architecture be under SC separation?



- Scalar Gaussian broadcast channel is degraded:
  - everything that receiver w/ $\downarrow$ SNR gets, the receiver with $\uparrow$ SNR gets
  - receiver w/ $\uparrow$ SNR can get extra info
- Source code construction should reflect this:
  - If source code sends only individual messages $S_{1\to 2}, S_{1\to 3}$ the ability of receiver w/ $\uparrow$ SNR to hear everything sent to the receiver w/ $\downarrow$ SNR is *wasted*
  - $\Rightarrow$ should use *multicast* messages! $S_{1\to\{2,3\}}, S_{1\to 2}, S_{1\to 2}$.

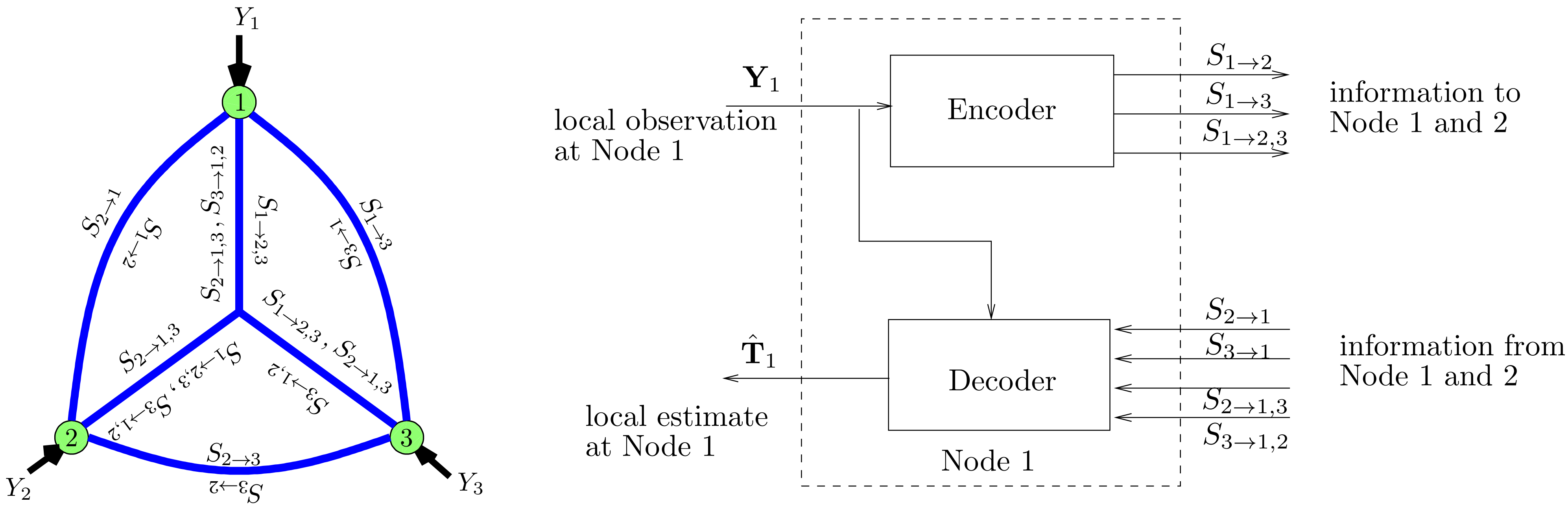
What should the source code architecture be?





- Network coding insight: limitation for $R_{1\to\{2,3\}}$ is 2, higher than maximum equal $R_{1\to 2}, R_{1\to 3} = \frac{3}{2}$.
- Again implies that (even separated) source coding construction should allow for *multicast* rates.

What should the source code architecture be?



- Each node sends a (possibly different nonempty) message to each subset of other nodes. $S_{j\to\mathcal{A}},\ \mathcal{A} \subseteq [M] \setminus j$.
- Every node collects all of the received messages together with its local observations and forms an estimate which minimizes its local Bayesian cost $\mathbb{E}\left[d_m(T, \hat{T}_m)\right]$.

What performance do the best such codes have? (Motivation)

- Rate distortion region $\mathcal{R}$ of achievable rate vector $\mathbf{r} := [R_{j\to\mathcal{A}} | j \in [M],\ \mathcal{A} \subseteq [M] \setminus j]$ and estimation error (cost) vector $\mathbf{d} := [D_j | j \in [M]]$ pairs characterizes the best such codes.

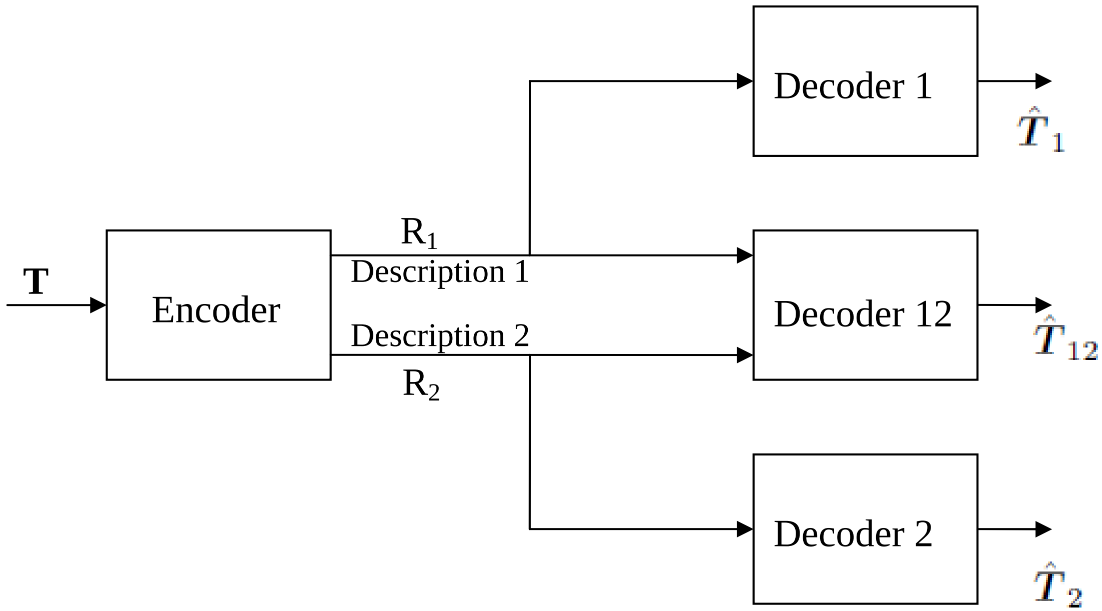
- Capacity region $\mathcal{C}$ of a network is described in terms of all achievable $\mathbf{r}$.

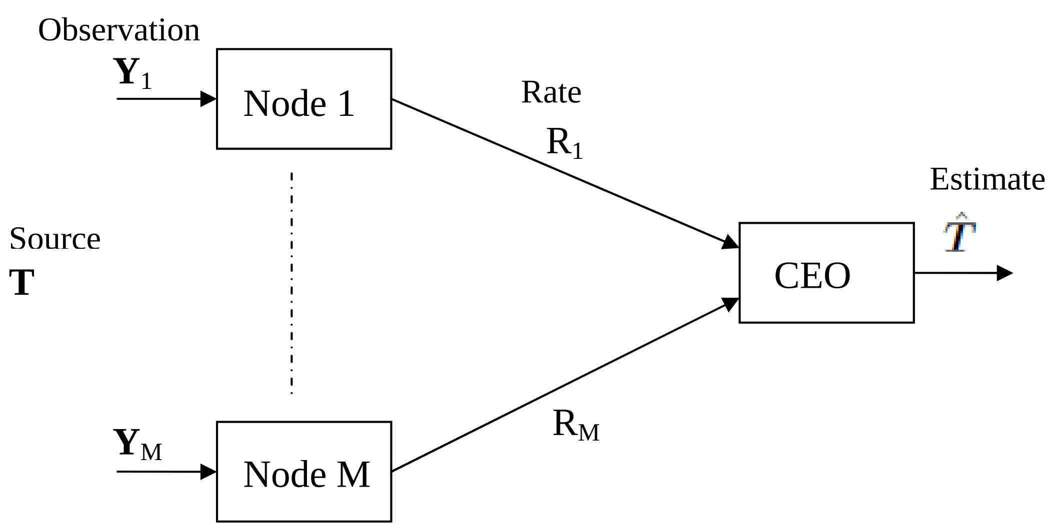
- Estimation performances attainable are those $\mathbf{d}$ associated with a $\mathbf{r}$ through $(\mathbf{r}, \mathbf{d}) \in \mathcal{R}$ with $\mathbf{r}$ in $\mathcal{C}$.

- Hence, inner and outer bounds for the rate distortion region $\mathcal{R}$ for this problem are of interest.

Rate Distortion Region

this problem is a hybrid btw. 2 classic incompletely solved IT problems...[19][20][21]





Rate Distortion Region: Inner Bound

- Multiple (M) Descriptions Achievability:
  1. select $p(\mathbf{U}|T)$ such that $\mathbb{E}[d(T, f(V, \{U_{\mathcal{A}}|\mathcal{A} \subseteq \mathcal{B}\}))] < D_{\mathcal{B}}$. Each element $U_{\mathcal{A}}$ of $\mathbf{U}$ corresponds to codeword avail. to nodes w/ all descriptions in $\mathcal{A} \subset [M]$.
  2. Generate codebook for $\mathcal{A}$ as $2^{N\tilde{R}_{\mathcal{A}}}$ length N codewords i.i.d. $p(U_{\mathcal{A}})$.

$$\sum_{\mathcal{A} \in \boldsymbol{\mathcal{P}}} \tilde{R}_{\mathcal{A}} > \sum_{\mathcal{A} \in \boldsymbol{\mathcal{P}}} H(U_{\mathcal{A}}|V) - H(\mathbf{U}_{\boldsymbol{\mathcal{P}}}|T, V) \quad \text{for all } \boldsymbol{\mathcal{P}} \subseteq 2^{[M]}$$

  makes sure $\exists$ codewords jointly typical w/ each other and T^N at encoder.

- CEO Achievability:
  1. select $p(U_i|Y_i)$, $U_{[M]\setminus i}, Y_{[M]\setminus i} \leftrightarrow Y_i V \leftrightarrow U_i$, such that $\mathbb{E}[d(T, f(U_{[M]}, V))] < D$.
  2. Generate codebook for i as $2^{N\tilde{R}_i}$ length N codewords iid $p(U_i)$ w/ $\tilde{R}_i > I(U_i; Y_i)$. Divide into 2^{NR_i} bins, send index of bin with codeword jointly typical with observations Y_i^N.

$$\sum_{i \in \mathcal{A}} R_i > I(Y_{\mathcal{A}}; U_{\mathcal{A}}|U_{[M]\setminus \mathcal{A}})$$

  makes sure $\exists$ codewords jointly typical w/ each other in bins at decoder.

Rate Distortion Region: Inner Bound

To hybridize the CEO and MD constructions, let each encoder in CEO encode multiple dependent descriptions, then bin. $\Longrightarrow$ both *encoder* (codebook size) & *decoder* (bin size) inequalities nontrivial. [22]

- $\mathcal{S}_i, \mathcal{D}_i$, messages sent, recvd at node i, resp.
- **Time Sharing**: V is independent from $\boldsymbol{Y}_{[M]}, T$
- **Encoding Constraints**: $T, \hat{\boldsymbol{T}}_{[M]}, \boldsymbol{Y}_{[M]\setminus i}, \boldsymbol{U}_{\mathcal{S}\setminus\mathcal{S}_i} \leftrightarrow Y_i, V \leftrightarrow \boldsymbol{U}_{\mathcal{S}_i}$
- **Decoding Constraints**: $T, \hat{\boldsymbol{T}}_{[M]\setminus i}, \boldsymbol{Y}_{[M]\setminus i}, \boldsymbol{U}_{\mathcal{S}\setminus\mathcal{D}_i} \leftrightarrow Y_i, \boldsymbol{U}_{\mathcal{D}_i}, V \leftrightarrow \hat{T}_i$, and $D_i > \mathbb{E}\left[d_i\left(T; \hat{T}_i\right)\right]$

Rate Distortion Region: Inner Bound

- **Codebooks:** $\forall\ \mathcal{P}_j \subseteq \mathcal{S}_j,\ \forall\ j \in [M]$

$$\sum_{(j\to\mathcal{A})\in\mathcal{P}_j} \tilde{R}_{j\to\mathcal{A}} > \sum_{(j\to\mathcal{A})\in\mathcal{P}_j} H(U_{j\to\mathcal{A}}|V) - H(\boldsymbol{U}_{\mathcal{P}_j}|Y_j, V),$$

Makes sure there is a collection of codewords in the codebooks jointly typical with each other and the observations at each *encoder.*

- **Bins:** for all $\mathcal{C}_i \subseteq \mathcal{D}_i$ and $i \in [M]$

$$\sum_{(j\to\mathcal{A})\in\mathcal{C}_i} R_{j\to\mathcal{A}} > \sum_{(j\to\mathcal{A})\in\mathcal{C}_i} \Big(\tilde{R}_{j\to\mathcal{A}} - H(U_{j\to\mathcal{A}}|V)\Big) + H(\boldsymbol{U}_{\mathcal{C}_i}|V, \boldsymbol{U}_{\mathcal{D}_i\setminus\mathcal{C}_i}, Y_i)$$

Makes sure that the bins are small enough such that there is only one collection of codewords jointly typical with each other and the side information at each *decoder*.

What is the major underlying fundamental (math) problem here?

- Major issue with these regions: while analytically elegant, it is difficult to determine whether or not a given $\mathbf{r}, \mathbf{d}$ pair lies within them.
- They involve inequalities among *rates* and (weighted) sums of *Shannon entropies* of subsets of random variables, including *auxiliary variables* (distribution not determined other than to obey certain distortion constraints).
- All rate regions in multiterminal information theory are expressible in this way.
- Hence all rate regions are expressed in terms of linear projections of $\bar{\Gamma}_N^*(\mathcal{C})$.
- Problem is, we don't know the boundaries of $\bar{\Gamma}_4^*$, let alone $\bar{\Gamma}_N^*$ or $\bar{\Gamma}_N^*(\mathcal{C})$.
- (research mentioned in the introduction)

How might these perspectives be reconciled?

Communication Network & Energy Constraints

low **high**

Computation & Delay Constraints

low

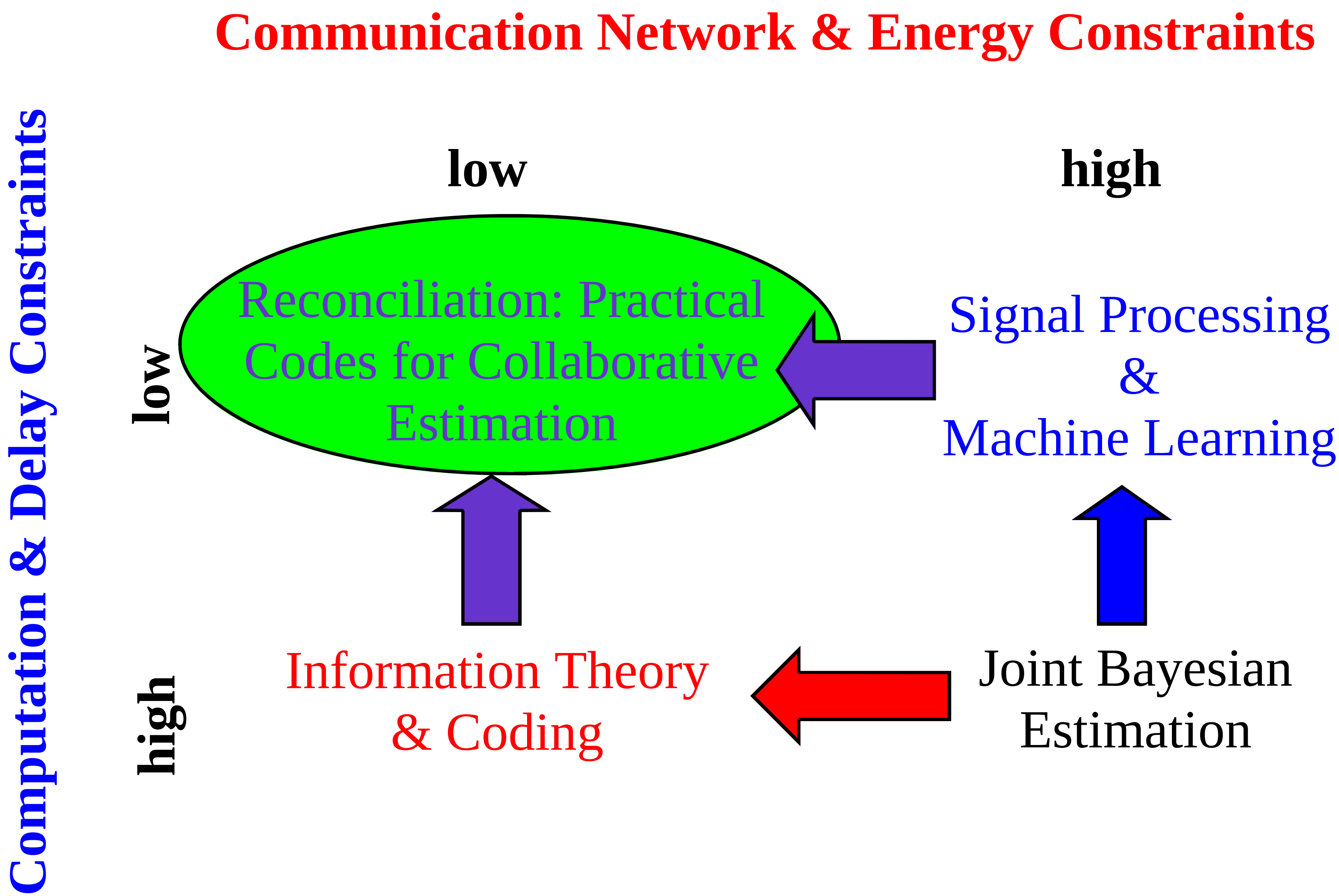
high

Signal Processing
&
Machine Learning

Information Theory
& Coding

Joint Bayesian
Estimation

How might these perspectives be reconciled?



How might these perspectives be reconciled?

- Sparse graph coding constructions and modifications of BP decoders have been adapted to some multiterminal coding problems (Wyner-Ziv, Slepian-Wolf)
- How can they be adapted and generalized to this one?
- What do the information theoretic bound evaluate to in important pragmatic estimation problems for wireless networks, such as for channel estimation?
- Belief/expectation propagation can help not only with designing the decoders, but also determining which information to compress in order to make risk minimization tractable after decoding.

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