Properties of the binary entropy vector region

John MacLaren Walsh and Steven Weber (Drexel University, Dept. of ECE, Philadelphia PA, 19104)

The entropy vector region (EVR) is the set of all entropic vectors for joint distributions with finite support.

- Let $X = (X_1, \ldots, X_n)$ be a discrete random variable with finite support.
- Let $H(X_k)$ be the entropy of the kth component of $X$.
- Let $H(X_k | X_{k-1}, \ldots, X_1)$ be the conditional entropy of the kth component given the previous k-1 components.

Theorem 1: The binary entropy vector region is a polygon with vertices at

- $H_2(X_1), H_2(X_2), \ldots, H_2(X_n)$
- $H_2(X_1|X_2), H_2(X_2|X_1), \ldots, H_2(X_n|X_{n-1})$

The Shannon inequalities give a polytope (polyhedron) containing the EVR.

- $H_2(X_1) + H_2(X_2) \leq H_2(X_1|X_2) + H_2(X_2|X_1)$
- $H_2(X_1) + H_2(X_2) \leq H_2(X_1|X_2) + H_2(X_2|X_1)$

The independent distributions map to the "top face" of the binary entropy region.

- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$
- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$

Fixed marginal distributions map to a "vertical" line of feasible joint entropies.

- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$
- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$

Intuition for inverse map for $n = 2$: Joint entropy can be expressed in terms of the marginal entropies.

- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$
- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$

Open question: characterization of which faces of $H_2(X_1) + H_2(X_2)$ are Shannon tight.

- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$
- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$

Intuition for inverse map for $n = 2$: Joint entropy can be expressed in terms of the marginal entropies.

- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$
- $H_2(X_1) + H_2(X_2) \leq H_2(X_1) + H_2(X_2)$

References