# Capacity region of the permutation channel

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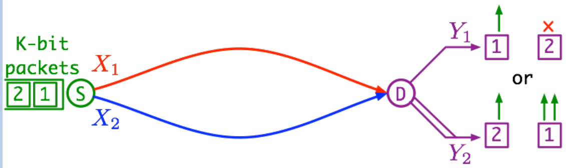
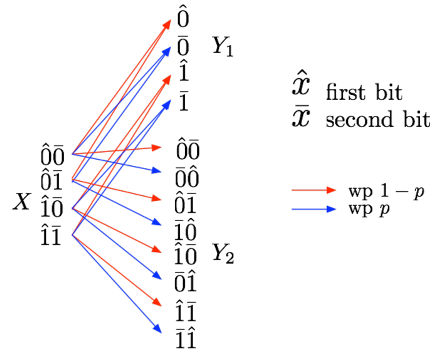
### How to use two paths to a destination?




- Minimize "delay": ensure content arrives in order by sending same content on both paths $\Rightarrow$ low throughput
- Maximize "throughput": send independent packets on the two paths $\Rightarrow$ higher delay
- Each packet reception instant is a receiver, rate tradeoff among receivers captures T-D tradeoff

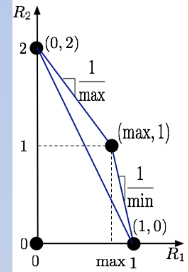


### Same problem, depicted as a broadcast channel





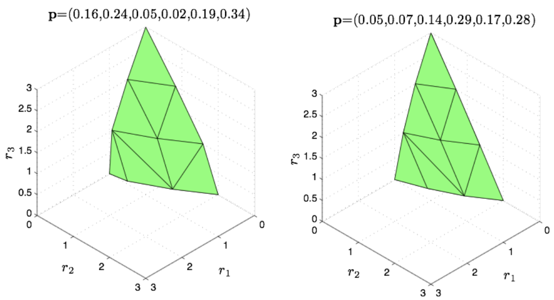

### Capacity region for two packets




- Increasing rate $R_1$ (decreasing playback delay) requires decrease in rate $R_2$ at slope $1/\max > 1$ or $1/\min > 1$
- Time-sharing inner bound achievable by single channel use
- Outside this region requires coding across multiple channel uses (moving away from capacity boundary increases reliability)

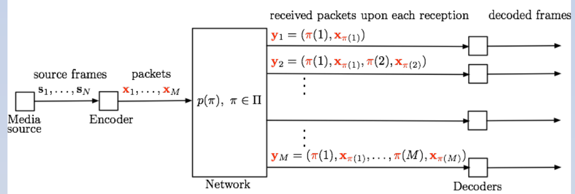


### Capacity region for three packets is convex hull of 10 rate points





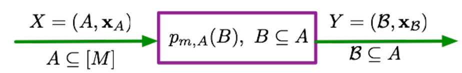

### Problem statement: sending $M$ packets over a permutation DBC




Goal: characterize the region of achievable rate points, $\mathcal{R}$, in terms of the specified permutation distribution $\mathbf{p} = (p(\pi), \pi \in \Pi)$.



### Achievability (inner bound): point to point channels




**Proposition:** $C(A) = K \sum_{\mathcal{B} \subseteq A} p_A(\mathcal{B})|\mathcal{B}| = K\mathbb{E}_A[|\mathcal{B}|]$.

**Proof.** Let $\mathcal{Q}$ be the collection of disbns $\mathbf{q}$ on the contents of packets $\mathbf{x}_A$.

$$C(A) \equiv \max_{\mathbf{q} \in \mathcal{Q}} H(Y) - H(Y|X)$$
$$= \max_{\mathbf{q} \in \mathcal{Q}} (H(\mathcal{B}) + H(\mathbf{x}_{\mathcal{B}}|\mathcal{B})) - (H(\mathcal{B})) = \max_{\mathbf{q} \in \mathcal{Q}} H(\mathbf{x}_{\mathcal{B}}|\mathcal{B})$$
$$= \max_{\mathbf{q} \in \mathcal{Q}} \sum_{\mathcal{B} \subseteq A} p_A(\mathcal{B}) H(\mathbf{x}_{\mathcal{B}}|\mathcal{B} = B) \leq \sum_{\mathcal{B} \subseteq A} p_A(\mathcal{B}) K|\mathcal{B}|$$

Equality is achieved by the uniform distribution $\mathbf{q}$.



### Achievability theorem (inner bound)

- Fix assignment $\mathbf{A} = (A_1, \ldots, A_M) \in \mathcal{A}_M$: consider $M$ P2P channels:

$$X_m \equiv (A_m, \mathbf{x}_{A_m}) \rightarrow Y_m \equiv (\mathcal{B}_m, \mathbf{x}_{\mathcal{B}_m}),\quad m \in [M]$$

- Achievable rate point for $\mathbf{A}$: $\mathbf{R}(\mathbf{A}) = (R_1(A_1), \ldots, R_M(A_M))$

$$R_m(A_m) = K \sum_{B \in \mathcal{S}(A_m, m)} p_{A_m}(B)|B|,\quad m \in [M]$$

- A region of achievable rates is defined by the polytope formed by the convex hull of the rate points for each possible packet assignment:

$$\mathcal{R}_M^{\text{in}} = \text{conv}\left(\{\mathbf{R}(\mathbf{A}),\ \mathbf{A} \in \mathcal{A}_M\}\right)$$

- Non-vertex points in this set are achievable via time-sharing across channel uses



### DBC capacity theorem as a linear map from mutual information vectors to rate points

- Define $u_m = (x_1, \ldots, x_m)$ and $y_m = (x_{\pi(1)}, \ldots, x_{\pi(m)})$
- General theorem for DBC with $M$ receivers:

$$R_m \leq I(y_m; u_m | u_{m-1}),\ m \in [M].$$

- Specialized to permutation channel with $M$ receivers:

$$R_m \leq \sum_{G \in \mathcal{G}_m} p_m(G) I(\mathbf{x}_G; u_m | u_{m-1}),\ m \in [M]$$

for appropriate support $\mathcal{G}_m$ and distribution $p_m(G)$, defined in terms of $(p(\pi), \pi \in \Pi)$.

- Capacity region $\mathcal{R} = \text{conv}(\{R_m\})$



### Entropy vectors through the linear map yields achievable rate points

- Convex hull of set of feasible mutual information vectors generated by a set of points $\{\mathbf{I}\}$
- Map each M.I. vector to a rate point: $\mathbf{R} = \mathbf{TI}$
- Form convex hull of these rate points $\mathcal{R}_M^{\text{out}} = \text{conv}(\{\mathbf{R}\})$

This procedure can be automated for any specified permutation distribution $(p(\pi), \pi \in \Pi)$.

### Summary

- **Theorem (inner bound):** $\mathcal{R}_M^{\text{in}} = \text{conv}(\{\mathbf{R}(\mathbf{A})\})$ for $\mathbf{A} \in \mathcal{A}_M$.
- **Theorem (outer bound):** $\mathcal{R}_M^{\text{out}} = \text{conv}(\{\mathbf{R}\})$ for $\mathbf{R} = \mathbf{TI}$.
- **Conjecture:** $\mathcal{R}_M^{\text{in}} = \mathcal{R}_M^{\text{out}}$. Verified for $M = 2, 3$.